Taking Limits of General Relativity

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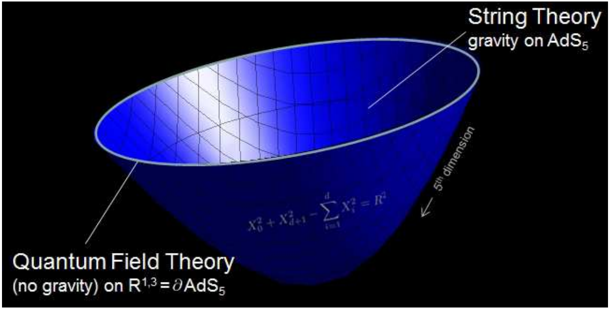
why non-relativistic gravity ?

The Holographic Principle



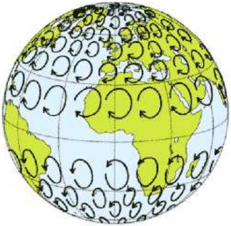
Gravity is not only used to describe the gravitational force!

Effective Field Theory

Examples: liquid helium, cold atomic gases and quantum Hall fluids

Effective Field Theory (EFT) coupled to NC gravity $\Rightarrow$ universal features

compare to



Coriolis force

Greiter, Wilczek, Witten (1989), Son (2005, 2012), Can, Laskin, Wiegmann (2014), Jensen (2014), Gromov, Abanov (2015)

Supersymmetry

supersymmetry allows to apply powerful localization techniques to exactly calculate partition functions of (non-relativistic) supersymmetric field theories

Pestun (2007); Festuccia, Seiberg (2011),

This should also apply to the non-relativistic case !

Non-relativistic Gravity

- Free-falling frames: Galilean symmetries

- Earth-based frame: Newtonian gravity/Newton potential $\Phi(x)$

- no frame-independent formulation (needs geometry!)

General Frames

- $\{\tau_\mu \,, e_\mu{}^a\} \quad a = 1, 2, 3; \mu = 0, 1, 2, 3$

- $\{\tau_\mu \,, e_\mu{}^a\}$ <u>and</u> m_μ

- 3D: $\{\tau_\mu \,, e_\mu{}^a\}$ <u>and</u> $m_\mu \,, s_\mu$

zero torsion : $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0 \quad \rightarrow \quad \tau_\mu = \partial_\mu \tau$

$$\tau(x) = t \quad \rightarrow \quad \tau_\mu = \delta_{\mu,0}$$

Take Home Message

Taking the non-relativistic limit is non-trivial and not unique !

Outline

Galilei Gravity : $\{\tau_\mu\,, e_\mu{}^a\}$

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Final Remarks

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Galilei Symmetries

- time translations : $\delta t = \xi^0$
- space translations : $\delta x^i = \xi^i \qquad i = 1, 2, 3$
- spatial rotations : $\delta x^i = \lambda^i{}_j \, x^j$
- Galilean boosts : $\delta x^i = \lambda^i t$

$$[J_{ab}, P_c] = -2\delta_{c[a}P_{b]} \,, \qquad [J_{ab}, G_c] = -2\delta_{c[a}G_{b]} \,,$$

$$[G_a, H] = -P_a \,, \qquad [J_{ab}, J_{cd}] = \delta_{c[a}J_{b]d} - \delta_{a[c}J_{d]b} \,, \qquad a = 1, 2, 3$$

'Gaugings', Contractions and Non-relativistic Limits

Poincare $\overset{\text{'gauging'}}{\Longrightarrow}$ General relativity

contraction $\Downarrow$ $\Downarrow$ non-relativistic limit

Galilei $\overset{\text{'gauging'}}{\Longrightarrow}$ Galilei Gravity

Inönü Wigner Contraction

$$\left[P_A, M_{BC}\right] = 2\,\eta_{A[B}\,P_{C]}\,, \qquad \left[M_{AB}, M_{CD}\right] = 4\,\eta_{[A[C}\,M_{D]B]}$$

$$P_0 = \frac{1}{2\omega}\,H\,, \qquad P_a = P_a\,, \qquad A = (0, a)$$

$$M_{ab} = J_{ab}\,, \qquad M_{a0} = \omega\,G_a$$

Taking the limit $\omega \to \infty$ gives the Galilei algebra:

$$\left[P_a, G_b\right] = 0$$

The Galilei Limit

Our starting point is the Einstein-Hilbert action in first-order formalism:

$$S = -\frac{1}{16\pi G_N}\int E E_A^\mu E_B^\nu R_{\mu\nu}{}^{AB}(M)$$

$$E_\mu^0 \;=\; \omega\tau_\mu\,, \qquad \Omega_\mu^{0a} \;=\; \omega^{-1}\omega_\mu^a\,, \qquad G_{\rm N} = \omega\, G_{\rm G} \qquad \Rightarrow$$

$$S_{\rm Gal} = -\frac{1}{16\pi G_{\rm G}}\int e\, e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab}(J)$$

accidental local scaling symmetry

$$\tau_\mu \rightarrow \lambda(x)^{-(D-3)}\tau_\mu\,, \qquad e_\mu{}^a \rightarrow \lambda(x) e_\mu{}^a$$

Constrained Geometry

For $D > 3$ the e.o.m. for $\omega_\mu{}^{ab}$ can be used to solve for $\omega_\mu{}^{ab}$

$$\omega_\mu{}^{ab} = \tau_\mu A^{ab} + e_{\mu c}\omega^{abc}(e, \tau)$$

except for an antisymmetric tensor component $A^{ab} = -A^{ba}$ of $\omega_\mu{}^{ab}$

Furthermore, the e.o.m. lead to the following restriction on the geometry:

$$\tau_{ab} \equiv e_a^\mu e_b^\nu \partial_{[\mu}\tau_{\nu]} = 0\ :\quad \text{twistless torsion} \qquad (e_a^\mu \tau_\mu = 0)$$

Using a second-order formalism the field A^{ab} acts as a Lagrange multiplier enforcing the constraint $\tau_{ab} = 0$

Carroll versus Galilei Gravity

Gomis, Rollier, Rosseel, ter Veldhuis + E.B. (2017)

Carroll gravity is the ultra-relativistic limit of Einstein gravity

The Carroll algebra is similar to but not the same as the Galilei algebra

- The Carroll action contains both a $R_{\mu\nu}{}^{ab}(J)$ and a $R_{\mu\nu}{}^{a}(G)$ term
- Symmetric Lagrange multiplier $S^{(ab)}$ and constraint $K_{(ab)} = 0$
- relation with strong coupling limit of Henneaux ?

Henneaux (1979)

Outline

Bargmann Symmetries

$$S_{\text{non-relativistic}}(\text{massive}) = \frac{m}{2}\int \frac{\dot{x}^i \dot{x}^j \delta_{ij}}{\dot{t}} d\tau \qquad i = 1, 2, 3$$

Lagrangian is <u>not</u> invariant under Galilean boosts $\delta\dot{x}^i = \lambda^i \dot{t}$:

$$\delta L_{\text{ non-relativistic}}(\text{massive}) = \frac{d}{d\tau}(m x^i \lambda^j \delta_{ij}) \qquad \Rightarrow$$

modified Noether charge gives rise to central extension:

$$\left[P_a, G_b\right] = \delta_{ab} Z$$

‘Gaugings’, Contractions and Non-relativistic Limits

Poincare $\otimes$ U(1) $\overset{\text{‘gauging’}}{\Longrightarrow}$ GR plus $\partial_\mu M_\nu - \partial_\nu M_\mu = 0$

contraction $\Downarrow$ $\Downarrow$ non-relativistic limit

Bargmann $\overset{\text{‘gauging’}}{\Longrightarrow}$ Newton-Cartan gravity

Inönü Wigner Contraction

$$[P_A, M_{BC}] = 2\,\eta_{A[B}\,P_{C]}\,, \qquad [M_{AB}, M_{CD}] = 4\,\eta_{[A[C}\,M_{D]B]} \qquad \text{plus} \qquad \mathcal{Z}$$

$$P_0 = \frac{1}{2\omega}\,H + \omega\,Z\,, \qquad \mathcal{Z} = \frac{1}{2\omega}\,H - \omega\,Z\,, \qquad A = (0, a)$$

$$P_a = P_a\,, \qquad M_{ab} = J_{ab}\,, \qquad M_{a0} = \omega\,G_a$$

Taking the limit $\omega \rightarrow \infty$ gives the Bargmann algebra including Z:

$$[P_a, G_b] = \delta_{ab}\,Z$$

The Newton-Cartan Limit I

Dautcourt (1964)

STEP I: express relativistic fields $\{E_\mu{}^A, M_\mu\}$ in terms of non-relativistic fields $\{\tau_\mu, e_\mu{}^a, m_\mu\}$

$$E_\mu{}^0 = \omega\,\tau_\mu + \frac{1}{2\omega}\,m_\mu\,, \quad M_\mu = \omega\,\tau_\mu - \frac{1}{2\omega}\,m_\mu\,, \quad E_\mu{}^a = e_\mu{}^a \quad \Rightarrow$$

$$E^\mu{}_a = e^\mu{}_a - \frac{1}{2\omega^2}\,\tau^\mu e^\rho{}_a m_\rho + \mathcal{O}\left(\omega^{-4}\right) \ \text{and similar for } E^\mu{}_0$$

The Newton-Cartan Limit II

STEP II: take the limit $\omega \to \infty$ in e.o.m. $\Rightarrow$

- the NC transformation rules are obtained
- the NC equations of motion are obtained (but no action!)

Note: the standard textbook limit gives Newton gravity

The NC Equations of Motion

The NC equations of motion are given by

$$\tau^\mu e^\nu{}_a \mathcal{R}_{\mu\nu}{}^a(G) = 0 \qquad \mathbf{1}$$

$$e^\nu{}_a \mathcal{R}_{\mu\nu}{}^{ab}(J) = 0 \qquad \mathbf{a} + (\mathbf{ab})$$

- after gauge-fixing and assuming flat space the first NC e.o.m. becomes $\triangle\Phi = 0$
- there is no known action that gives rise to these equations of motion

Coupling Newton-Cartan to Matter

Jensen, Karch (2014), Fuini, Karch, Uhlemann (2015)

matter couplings (without torsion) from arbitrary contracting backgrounds

Rosseel, Zojer + E.B. (2015)

Klein-Gordon + GR $\stackrel{\text{`limit'}}{\Longrightarrow}$ Schrödinger + NC

general frames $\Uparrow$ $\Downarrow$ free-falling frames

Klein-Gordon $\stackrel{?}{\Longrightarrow}$ Schrödinger

From Klein-Gordon to Schrödinger I

we consider a complex scalar field with mass M

Lévy Leblond (1963,1967)

$$E^{-1}\,\mathcal{L}_{\rm rel} = -\frac{1}{2}\,g^{\mu\nu}\,D_\mu\Phi^*\,D_\nu\Phi - \frac{M^2}{2}\,\Phi^*\Phi \quad \text{with}$$

$$D_\mu\Phi = \partial_\mu\Phi - {\rm i}\,M\,M_\mu\,\Phi\,, \qquad\qquad \delta\Phi = {\rm i}\,M\,\Lambda\,\Phi$$

- M_μ is not an electromagnetic field ($M \neq q$)!
- M_μ couples to the current that expresses conservation of
  # particles – # antiparticles
- going to free-falling frames gives Klein-Gordon

From Klein-Gordon to Schrödinger II

Take non-relativistic limit extended with $M = \omega m\, , \Phi = \sqrt{\frac{\omega}{m}}\phi \quad \rightarrow$

$$e^{-1}\mathcal{L}_{\text{Schroedinger}} = \Big[\, \frac{\mathrm{i}}{2}\Big(\phi^* \tilde{D}_0\phi - \phi\tilde{D}_0\phi^*\Big) - \frac{1}{2m}\Big|\tilde{D}_a\phi\Big|^2\,\Big] \quad \text{with}$$

$$\tilde{D}_\mu\phi = \partial_\mu\phi + \mathrm{i}\, m\, m_\mu\, \phi\, , \qquad \delta\phi = \xi^\mu\partial_\mu\phi - \mathrm{i}\, m\,\sigma\,\phi$$

- m_μ couples to the current that expresses conservation of # particles
- going to free-falling frames gives Schrödinger

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Extended Bargmann Symmetries

Papageorgiou, Schroers (2009); Rosseel + E.B. (2016); Hartong, Lei, Obers (2016)

Galilei $\overset{\text{`Mass'}}{\Longrightarrow}$ Bargmann $\overset{\text{`Spin'}}{\Longrightarrow}$ Extended Bargmann

Lévy-Leblond (1972), Jackiw, Nair (2000)

$$[J_A, P_B] = -\epsilon_{ABC}P^C\,, \qquad [J_A, J_B] = -\epsilon_{ABC}J^C \qquad \text{plus} \qquad \mathcal{Z}_1\,,\mathcal{Z}_2$$

$$[H, G_a] = -\epsilon_{ab}P_b\,, \qquad [J, G_a] = -\epsilon_{ab}G_b\,, \qquad [J, P_a] = -\epsilon_{ab}P_b\,,$$

$$[G_a, P_b] = \epsilon_{ab}M\,, \qquad [G_a, G_b] = \epsilon_{ab}S$$

The 3D Extended Bargmann Limit

$$S = \frac{k}{4\pi}\int d^3x\,\left(\epsilon^{\mu\nu\rho}E_\mu{}^A R^A_{\nu\rho}(J) + 2\epsilon^{\mu\nu\rho}Z_{1\mu}\partial_\nu Z_{2\rho}\right)$$

Einstein + extra term

$$E_\mu{}^0 = \omega\tau_\mu + \frac{1}{2\omega}m_\mu\,, \qquad Z_{1\mu} = \omega\tau_\mu - \frac{1}{2\omega}m_\mu$$

$$\Omega_\mu{}^0 = \omega\tau_\mu + \frac{1}{2\omega^2}s_\mu\,, \qquad Z_{2\mu} = \omega\tau_\mu - \frac{1}{2\omega^2}s_\mu$$

$$E_\mu{}^a = e_\mu{}^a\,, \qquad \Omega_\mu{}^a = \frac{1}{\omega}\omega_\mu{}^a$$

plus $k \to k\omega$

3D Extended Bargmann Gravity

3D extended Bargmann has invariant, non-degenerate bilinear form:

$$< J_a, P_b >= \delta_{ab}\,, \qquad < M, J >= -1\,, \qquad < H, S >= -1 \qquad \Rightarrow$$

$$S = \frac{k}{4\pi}\int d^3x \left(\epsilon^{\mu\nu\rho} e_\mu{}^a R_{\nu\rho}{}^a(G) - \epsilon^{\mu\nu\rho} m_\mu R_{\nu\rho}(J) - \epsilon^{\mu\nu\rho}\tau_\mu R_{\nu\rho}(S)\right)$$

- more general curved background solutions than Newton Cartan
- SUSY extension exists

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Condensed Matter Physics

Use Effective Field Theory (EFT)

Son, Wingate (2006)

Gravitational response gives information about geometric quantities such as the Hall viscosity

Coupling NC gravity to EFT leads to less free parameters than physical quantities $\Rightarrow$

Relation between Hall conductivity and Hall viscosity

Hoyos, Son (2012)

March 6-10, 2017: Save the Date!

Simons Workshop on Applied Newton-Cartan Geometry

organized by Gary Gibbons, Rob Leigh, Djordje Minic, Dam Thanh Son + E.B.