Spin(7)-instantons & other Yang-Mills solutions on cylinders over coset spaces with $G_2$-structure

Alexander Haupt
University of Hamburg

11th Nordic String Theory Meeting 2017, Hannover
10-Feb-2017

JHEP 1603(2016)038 & WIP
Outline

1. Introduction
   - Motivation
   - Yang-Mills instantons in \( d = 4 \)
   - Instantons in \( d > 4 \) & YM with torsion

2. YM theory & instantons on 8d \( Z(G/H) \)
   - Quick review of 7d \( G_2 \) & 8d Spin(7)-structures
   - Set-up: gauge field ansatz
   - Solutions: old & new

3. Conclusions
1 Introduction
   - Motivation
   - Yang-Mills instantons in $d = 4$
   - Instantons in $d > 4$ & YM with torsion

2 YM theory & instantons on 8d $Z(G/H)$
   - Quick review of 7d $G_2$- & 8d Spin(7)-structures
   - Set-up: gauge field ansatz
   - Solutions: old & new

3 Conclusions
1 Introduction
   - Motivation
   - Yang-Mills instantons in $d = 4$
   - Instantons in $d > 4$ & YM with torsion

2 YM theory & instantons on 8d $\mathbb{Z}(G/H)$
   - Quick review of 7d $G_2$- & 8d Spin(7)-structures
   - Set-up: gauge field ansatz
   - Solutions: old & new

3 Conclusions
In the **low-energy limit**, heterotic string theory yields $\mathcal{N} = 1$, $d = 10$ supergravity coupled to super Yang-Mills theory

- In phenomenological applications, one often considers “string compactifications”: $M^{10} = M^{10-n} \times X_n$

- Of particular interest are solutions that preserve some amount of supersymmetry

- Condition of SUSY preservation leads to appearance of higher-dim. Yang-Mills instantons and $G$-structure manifolds

**Overarching aim(s):**

1. **construct new instanton/YM solutions on various $G$-structure manifolds**

2. **find embeddings into string theory (het. SUGRA)**
In the **low-energy limit**, heterotic string theory yields $\mathcal{N} = 1, d = 10$ supergravity coupled to super Yang-Mills theory.

In phenomenological applications, one often considers "**string compactifications**": $\mathcal{M}^{10} = \mathcal{M}^{10-n} \times X_n$

Of particular interest are solutions that preserve some amount of supersymmetry.

Condition of SUSY preservation leads to appearance of higher-dim. **YM-instantons and $G$-structure manifolds**

**Overarching aim(s):**

1. **Construct new instanton/YM solutions on various $G$-structure manifolds**
2. **Find embeddings into string theory (het. SUGRA)**
In the **low-energy limit**, heterotic string theory yields $\mathcal{N} = 1$, $d = 10$ supergravity coupled to super Yang-Mills theory.

In phenomenological applications, one often considers “**string compactifications**”: $\mathcal{M}^{10} = \mathcal{M}^{10-n} \times X_n$.

Of particular interest are solutions that preserve some amount of **supersymmetry**.

Condition of SUSY preservation leads to appearance of higher-dim. **YM-instantons and G-structure manifolds**.

**Overarching aim(s):**

1. **Construct new instanton/YM solutions on various G-structure manifolds**
2. **Find embeddings into string theory (het. SUGRA)**
In the **low-energy limit**, heterotic string theory yields $\mathcal{N} = 1, d = 10$ supergravity coupled to super Yang-Mills theory.

In phenomenological applications, one often considers “string compactifications”: $\mathcal{M}^{10} = \mathcal{M}^{10-n} \times X_n$.

Of particular interest are solutions that preserve some amount of **supersymmetry**.

Condition of SUSY preservation leads to appearance of higher-dim. **YM-instantons and $G$-structure manifolds**.

**Overarching aim(s):**

1. construct new instanton/YM solutions on various $G$-structure manifolds
2. find embeddings into string theory (het. SUGRA)
In the **low-energy limit**, heterotic string theory yields $\mathcal{N} = 1$, $d = 10$ supergravity coupled to super Yang-Mills theory

In phenomenological applications, one often considers “**string compactifications**”: $\mathcal{M}^{10} = \mathcal{M}^{10-n} \times X_n$

Of particular interest are solutions that preserve some amount of **supersymmetry**

Condition of SUSY preservation leads to appearance of higher-dim. **YM-instantons and $G$-structure manifolds**

**Overarching aim(s):**

1. **construct new instanton/YM solutions on various $G$-structure manifolds**

2. **find embeddings into string theory (het. SUGRA)**
In the **low-energy limit**, heterotic string theory yields $\mathcal{N} = 1$, $d = 10$ supergravity coupled to super Yang-Mills theory.

In phenomenological applications, one often considers “**string compactifications**”: $\mathcal{M}^{10} = \mathcal{M}^{10-n} \times X_n$

Of particular interest are solutions that preserve some amount of **supersymmetry**.

Condition of SUSY preservation leads to appearance of higher-dim. **YM-instantons and $G$-structure manifolds**

**Overarching aim(s):**

1. **Construct new instanton/YM solutions on various $G$-structure manifolds**
2. **Find embeddings into string theory (het. SUGRA)**
Definition

A Yang-Mills instanton is a gauge connection\(^\ast\) on Euclidean \(\mathcal{M}^4\), whose curvature \(F\) is self-dual, i.e. \(\ast F = F\).

\(^\ast\)connection \(A\nabla\) on a principal \(K\)-bundle over \(\mathcal{M}^4\) (gauge group \(K\))

[Belavin, Polyakov, Schwarz, Tyupkin (1975); Atiyah, Hitchin, Singer (1977); Atiyah, Drinfeld, Hitchin, Manin (1977), ...]

Properties

- Solutions of YM-eq. \((0 \overset{\text{BI}}{=} DF = D \ast F \implies D \ast F = 0)\)
- 1st order eq. easier to solve than 2nd order YM-eq.
- 1st ex: **BPST instanton** (1975) for \(\mathcal{M} = \mathbb{R}^4, K = SU(2)\)

Widespread applications in maths & physics

- classification of 4-manifolds (e.g. Donaldson invariants)
- learn about **structure of YM-vacuum** (crit. pts. of YM-action; appear in path int. as leading qu. corr.)
Definition

A Yang-Mills instanton is a gauge connection\(^{(*)}\) on Euclidean \(M^4\), whose curvature \(F\) is self-dual, i.e. \(*F = F\).

\(^{(*)}\)connection \(A\nabla\) on a principal \(K\)-bundle over \(M^4\) (gauge group \(K\))

[Belavin, Polyakov, Schwarz, Tyupkin (1975); Atiyah, Hitchin, Singer (1977); Atiyah, Drinfeld, Hitchin, Manin (1977), . . .]

Properties

- Solutions of YM-eq. \((0 \overset{\text{Bl}}{=} DF = D*F \implies D*F = 0)\)
- 1st order eq. easier to solve than 2nd order YM-eq.
- 1st ex: **BPST instanton** (1975) for \(M = \mathbb{R}^4, K = SU(2)\)

Widespread applications in maths & physics

- classification of 4-manifolds (e.g. Donaldson invariants)
- learn about structure of YM-vacuum (crit. pts. of YM-action; appear in path int. as leading qu. corr.)
Definition

A Yang-Mills instanton is a gauge connection*) on Euclidean $\mathcal{M}^4$, whose curvature $F$ is self-dual, i.e. $\star F = F$.

*)connection $^A \nabla$ on a principal $K$-bundle over $\mathcal{M}^4$ (gauge group $K$)

[Belavin, Polyakov, Schwarz, Tyupkin (1975); Atiyah, Hitchin, Singer (1977); Atiyah, Drinfeld, Hitchin, Manin (1977), ...]

Properties

- Solutions of YM-eq. $(0 \overset{\text{BI}}{=} DF = D \star F \implies D \star F = 0)$
- 1st order eq. easier to solve than 2nd order YM-eq.
- 1st ex: **BPST instanton** (1975) for $\mathcal{M} = \mathbb{R}^4$, $K = SU(2)$

Widespread applications in maths & physics

- classification of 4-manifolds (e.g. Donaldson invariants)
- learn about structure of YM-vacuum (crit. pts. of YM-action; appear in path int. as leading qu. corr.)
Definition

In higher dimensions, the instanton equation is generalized to

\[ \ast F = -F \wedge \ast Q_M , \]

with some globally well-defined 4-form \( Q_M \).

[Corrigan, Devchand, Fairlie, Nuyts (1983); Ward (1984), ...]

Properties

- Need additional structure on \( M \) to have \( Q_M \leftrightarrow G \)-structure manifolds (i.e. struct. grp. \( G \subset SO(d) \), e.g. \( SU(3) \) in \( d = 6 \))
- Instanton eq. \( \implies \) YM with torsion \( D \ast F + F \wedge \ast H = 0 \). Torsion 3-form \( \ast H := d \ast Q_M \) (ordinary YM if \( Q_M \) co-closed).

\( H \) appears naturally in string theory (curvature of NS 2-form)

Alternative defs (in many phys. appl.: 3 defs. equivalent)

- \( F \cdot \epsilon = 0 \) (BPS eq. in string theory)
- \( F \in g \) (i.e. \( F \in \Gamma(gM \otimes \text{End}(E)) \), often in math. lit)
**Definition**

In higher dimensions, the **instanton equation** is generalized to

\[ \ast F = - F \wedge \ast Q_M, \]

with some globally well-defined 4-form \( Q_M \).

[Corrigan, Devchand, Fairlie, Nuyts (1983); Ward (1984), ...]

**Properties**

- Need additional structure on \( M \) to have \( Q_M \leftrightarrow G\)-structure manifolds (i.e. struct. grp. \( G \subset SO(d) \), e.g. \( SU(3) \) in \( d = 6 \))
- Instanton eq. \( \implies \) **YM with torsion** \( D \ast F + F \wedge \ast H = 0 \).
  
  Torsion 3-form \( \ast H := d \ast Q_M \) (ordinary YM if \( Q_M \) co-closed).
  
  \( H \) appears naturally in string theory (curvature of NS 2-form)

**Alternative defs** (in many phys. applic.: 3 defs. equivalent)

- \( F \cdot \epsilon = 0 \) (BPS eq. in string theory)
- \( F \in g \) (i.e. \( F \in \Gamma(\mathfrak{g} M \otimes \text{End}(E)) \), often in math. lit)
Definition

In higher dimensions, the **instanton equation** is generalized to

\[ *F = -F \wedge *Q_M , \]

with some globally well-defined 4-form \( Q_M \).

[Corrigan, Devchand, Fairlie, Nuyts (1983); Ward (1984), ...]

Properties

- Need additional structure on \( M \) to have \( Q_M \leftrightarrow G\text{-structure manifolds} \) (i.e. struct. grp. \( G \subset SO(d) \), e.g. \( SU(3) \) in \( d = 6 \))
- Instanton eq. \( \implies \text{YM with torsion} \quad D * F + F \wedge *H = 0. \)
  
  Torsion 3-form \( *H := d * Q_M \) (ordinary YM if \( Q_M \) co-closed).
  
  \( H \) appears naturally in string theory (curvature of NS 2-form)

Alternative defs (in many phys. applic.: 3 defs. equivalent)

- \( F \cdot \epsilon = 0 \) (BPS eq. in string theory)
- \( F \in g \) (i.e. \( F \in \Gamma(gM \otimes \text{End}(E)) \), often in math. lit)
Scope of rest of talk

Find explicit solutions of instanton eq. and YM-eq. w/ torsion on “cylinder” \( \mathbb{Z}(G/H) := \mathbb{R} \times G/H \).

- \( G/H \) is a 7d compact coset space w/ \( G_2 \)- or \( SU(3) \)-structure
- Cylinder metric: \( g = d\tau \otimes d\tau + \delta_{ab} e^a \otimes e^b \) (\( a, b = 1, \ldots, 7 \))
- \( \{e^\mu\} = \{e^0 = d\tau, e^a\} \) is a local ONB of \( T^*(\mathbb{R} \times G/H) \)
- Why coset spaces? → simple non-triv. examples of \( G \)-structure manifolds (e.g. manageable)
- Why cylinders? → reduce to ODEs (gradient flow eqs.) in \( \tau \)
- Further motivation
  - Soln in gauge sector of heterotic flux compactifications
    (as e.g. in [AH, Lechtenfeld, Musaev (2014)])
  - Fill a gap in literature on higher-dim YM instantons
    [Lechtenfeld, Bauer, Bunk, Geipel, Gemmer, Harland, Ivanova, Lubbe, Nölle, Popov, Rahn, Sperling, Tormählen, AH, ... (2009–...)]
Find explicit solutions of instanton eq. and YM-eq. w/ torsion on “cylinder” $Z(G/H) := \mathbb{R} \times G/H$.

- $G/H$ is a 7d compact coset space w/ $G_2$- or $SU(3)$-structure
- Cylinder metric: $g = d\tau \otimes d\tau + \delta_{ab} e^a \otimes e^b$ ($a, b = 1, \ldots, 7$)
- $\{e^\mu\} = \{e^0 = d\tau, e^a\}$ is a local ONB of $T^*(\mathbb{R} \times G/H)$
- Why coset spaces? → simple non-triv. examples of $G$-structure manifolds (eqs. manageable)
- Why cylinders? → reduce to ODEs (gradient flow eqs.) in $\tau$
- Further motivation
  - Soln in gauge sector of heterotic flux compactifications (as e.g. in [AH, Lechtenfeld, Musaev (2014)])
  - **Fill a gap** in literature on higher-dim YM instantons
    [Lechtenfeld, Bauer, Bunk, Geipel, Gemmer, Harland, Ivanova, Lubbe, Nölle, Popov, Rahn, Sperling, Tormählen, AH, ... (2009–...)]
Scope of rest of talk

Find explicit solutions of instanton eq. and YM-eq. w/ torsion on “cylinder” \( Z(G/H) := \mathbb{R} \times G/H \).

- \( G/H \) is a 7d compact coset space w/ \( G_2 \)- or \( SU(3) \)-structure
- Cylinder metric: \( g = d\tau \otimes d\tau + \delta_{ab} e^a \otimes e^b \) \( (a, b = 1, \ldots, 7) \)
- \( \{ e^\mu \} = \{ e^0 = d\tau, e^a \} \) is a local ONB of \( T^*(\mathbb{R} \times G/H) \)
- Why coset spaces? → simple non-triv. examples of \( G \)-structure manifolds (eqs. manageable)
- Why cylinders? → reduce to ODEs (gradient flow eqs.) in \( \tau \)
- Further motivation
  - Soln in gauge sector of heterotic flux compactifications
    (as e.g. in [AH, Lechtenfeld, Musaev (2014)])
  - Fill a gap in literature on higher-dim YM instantons
    [Lechtenfeld, Bauer, Bunk, Geipel, Gemmer, Harland, Ivanova, Lubbe, Nölle, Popov, Rahn, Sperling, Tormählen, AH, ... (2009–...)]
Scope of rest of talk

Find explicit solutions of instanton eq. and YM-eq. w/ torsion on “cylinder” \( Z(G/H) := \mathbb{R} \times G/H \).

- \( G/H \) is a 7d compact coset space w/ \( G_2 \)- or \( SU(3) \)-structure
- Cylinder metric: \( g = d\tau \otimes d\tau + \delta_{ab} e^a \otimes e^b \) \((a, b = 1, \ldots, 7)\)
- \( \{e^\mu\} = \{e^0 = d\tau, e^a\} \) is a local ONB of \( T^*(\mathbb{R} \times G/H) \)
- Why coset spaces? → simple non-triv. examples of \( G \)-structure manifolds (eqs. manageable)
- Why cylinders? → reduce to ODEs (gradient flow eqs.) in \( \tau \)
- Further motivation
  - Soln in gauge sector of heterotic flux compactifications (as e.g. in [AH, Lechtenfeld, Musaev (2014)])
  - Fill a gap in literature on higher-dim YM instantons [Lechtenfeld, Bauer, Bunk, Geipel, Gemmer, Harland, Ivanova, Lubbe, Nölle, Popov, Rahn, Sperling, Tormählen, AH, ... (2009–...)]
Scope of rest of talk

Find explicit solutions of instanton eq. and YM-eq. w/ torsion on “cylinder” $Z(G/H) := \mathbb{R} \times G/H$.

- $G/H$ is a 7d compact coset space w/ $G_2$- or $SU(3)$-structure
- Cylinder metric: $g = d\tau \otimes d\tau + \delta_{ab} e^a \otimes e^b$ ($a, b = 1, \ldots, 7$)
- $\{e^\mu\} = \{e^0 = d\tau, e^a\}$ is a local ONB of $T^*(\mathbb{R} \times G/H)$
- Why coset spaces? → simple non-triv. examples of $G$-structure manifolds (eqs. manageable)
- Why cylinders? → reduce to ODEs (gradient flow eqs.) in $\tau$
- Further motivation
  - Soln in gauge sector of heterotic flux compactifications (as e.g. in [AH, Lechtenfeld, Musaev (2014)])
  - Fill a gap in literature on higher-dim YM instantons
    [Lechtenfeld, Bauer, Bunk, Geipel, Gemmer, Harland, Ivanova, Lubbe, Nölle, Popov, Rahn, Sperling, Tormählen, AH, ... (2009–...)]
Scope of rest of talk

Find explicit solutions of instanton eq. and YM-eq. w/ torsion on "cylinder" $Z(G/H) := \mathbb{R} \times G/H$.

- $G/H$ is a 7d compact coset space w/ $G_2$- or $SU(3)$-structure
- Cylinder metric: $g = d\tau \otimes d\tau + \delta_{ab} e^a \otimes e^b \ (a, b = 1, \ldots, 7)$
- $\{e^\mu\} = \{e^0 = d\tau, e^a\}$ is a local ONB of $T^*(\mathbb{R} \times G/H)$
- **Why coset spaces?** → simple non-triv. examples of $G$-structure manifolds (eqs. manageable)
- **Why cylinders?** → reduce to ODEs (gradient flow eqs.) in $\tau$
- Further motivation
  - Soln in gauge sector of heterotic flux compactifications (as e.g. in [AH, Lechtenfeld, Musaev (2014)])
  - **Fill a gap** in literature on higher-dim YM instantons [Lechtenfeld, Bauer, Bunk, Geipel, Gemmer, Harland, Ivanova, Lubbe, Nölle, Popov, Rahn, Sperling, Tormählen, AH, ... (2009–...)]
Scope of rest of talk

Find explicit solutions of instanton eq. and YM-eq. w/ torsion on “cylinder” \( Z(G/H) := \mathbb{R} \times G/H \).

- \( G/H \) is a 7d compact coset space w/ \( G_2 \)- or \( SU(3) \)-structure
- Cylinder metric: \( g = d\tau \otimes d\tau + \delta_{ab} e^a \otimes e^b \) \( (a, b = 1, \ldots, 7) \)
- \( \{e^\mu\} = \{e^0 = d\tau, e^a\} \) is a local ONB of \( T^*(\mathbb{R} \times G/H) \)
- Why coset spaces? \( \rightarrow \) simple non-triv. examples of \( G \)-structure manifolds (eqs. manageable)
- Why cylinders? \( \rightarrow \) reduce to ODEs (gradient flow eqs.) in \( \tau \)
- Further motivation
  - Soln in gauge sector of heterotic flux compactifications
    (as e.g. in [AH, Lechtenfeld, Musaev (2014)])
  - Fill a gap in literature on higher-dim YM instantons
    [Lechtenfeld, Bauer, Bunk, Geipel, Gemmer, Harland, Ivanova, Lubbe, Nölle, Popov, Rahn, Sperling, Tormählen, AH, ... (2009–...)]
7d $G_2$-structures:

- $G_2$-str. def. by **3-form** $P$ (Hodge dual **4-form** $Q := \ast_7 P$)
- $G_2$-structures distinguished/classified by **4 torsion classes**:

$$dP = \tau_0 Q + 3 \tau_1 \wedge P + \ast_7 \tau_3 , \quad dQ = 4 \tau_1 \wedge Q + \tau_2 \wedge P$$

- Important examples:

<table>
<thead>
<tr>
<th>Type</th>
<th>TCs</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallel</td>
<td>$\emptyset$</td>
<td>$dP = 0, dQ = 0$</td>
</tr>
<tr>
<td>nearly parallel</td>
<td>$\tau_0$</td>
<td>$dP = \tau_0 Q, dQ = 0$</td>
</tr>
<tr>
<td>cocalibrated/semi-p.</td>
<td>$\tau_0 \oplus \tau_3$</td>
<td>$dP = \tau_0 Q + \ast_7 \tau_3, dQ = 0$</td>
</tr>
</tbody>
</table>

8d Spin(7)-structures:

- $Z(G/H)$ inherits Spin(7)-str. def. by **self-dual 4-form** $\Psi$

$$\Psi = P \wedge d\tau - Q$$

- Spin(7)-structures distinguished by **2 torsion classes**

- **Dictionary**: 7d $G_2$-structures $\leftrightarrow$ Spin(7)-structures on cyl.
  e.g. 7d loc. conf. $G_2$-str. $\rightarrow$ 8d loc. conf. Spin(7)-str. on cyl.
7d $G_2$-structures:
- $G_2$-str. def. by **3-form** $P$ (Hodge dual **4-form** $Q := \ast_7 P$)
- $G_2$-structures distinguished/classified by **4 torsion classes**:
  \[ dP = \tau_0 Q + 3 \tau_1 \wedge P + \ast_7 \tau_3, \quad dQ = 4 \tau_1 \wedge Q + \tau_2 \wedge P \]
- **Important examples**:

<table>
<thead>
<tr>
<th>Type</th>
<th>TCs</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallel</td>
<td>$\emptyset$</td>
<td>$dP = 0, dQ = 0$</td>
</tr>
<tr>
<td>nearly parallel</td>
<td>$\tau_0$</td>
<td>$dP = \tau_0 Q, dQ = 0$</td>
</tr>
<tr>
<td>cocalibrated/semi-p.</td>
<td>$\tau_0 \oplus \tau_3$</td>
<td>$dP = \tau_0 Q + \ast_7 \tau_3, dQ = 0$</td>
</tr>
</tbody>
</table>

8d Spin(7)-structures:
- $Z(G/H)$ inherits Spin(7)-str. def. by **self-dual 4-form** $\Psi$
  \[ \Psi = P \wedge d\tau - Q \]
- Spin(7)-structures distinguished by **2 torsion classes**
- **Dictionary**: 7d $G_2$-structures $\leftrightarrow$ Spin(7)-structures on cyl.
  e.g. 7d loc. conf. $G_2$-str. $\rightarrow$ 8d loc. conf. Spin(7)-str. on cyl.
7d $G_2$-structures:

- $G_2$-str. def. by **3-form** $P$ (Hodge dual **4-form** $Q := \ast_7 P$)
- $G_2$-structures distinguished/classified by **4 torsion classes**:
  \[ dP = \tau_0 Q + 3\tau_1 \wedge P + \ast_7 \tau_3, \quad dQ = 4\tau_1 \wedge Q + \tau_2 \wedge P \]
- Important examples:

<table>
<thead>
<tr>
<th>Type</th>
<th>TCs</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallel</td>
<td>$\emptyset$</td>
<td>$dP = 0, dQ = 0$</td>
</tr>
<tr>
<td>nearly parallel</td>
<td>$\tau_0$</td>
<td>$dP = \tau_0 Q, dQ = 0$</td>
</tr>
<tr>
<td>cocalibrated/semi-p.</td>
<td>$\tau_0 \oplus \tau_3$</td>
<td>$dP = \tau_0 Q + \ast_7 \tau_3, dQ = 0$</td>
</tr>
</tbody>
</table>

8d Spin(7)-structures:

- $Z(\mathbb{G}/H)$ inherits Spin(7)-str. def. by self-dual **4-form** $\Psi$
  \[ \Psi = P \wedge d\tau - Q \]
- Spin(7)-structures distinguished by **2 torsion classes**
- **Dictionary**: 7d $G_2$-structures $\leftrightarrow$ Spin(7)-structures on cyl.
  e.g. 7d loc. conf. $G_2$-str. $\rightarrow$ 8d loc. conf. Spin(7)-str. on cyl.
7d $G_2$-structures:

- $G_2$-str. def. by **3-form $P$** (Hodge dual **4-form $Q := *_7 P$**)
- $G_2$-structures distinguished/classified by **4 torsion classes**:
  \[
  dP = \tau_0 \, Q + 3 \, \tau_1 \wedge P + *_7 \tau_3 , \quad dQ = 4 \, \tau_1 \wedge Q + \tau_2 \wedge P
  \]
- Important examples:

<table>
<thead>
<tr>
<th>Type</th>
<th>TCs</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallel</td>
<td>$\emptyset$</td>
<td>$dP = 0$, $dQ = 0$</td>
</tr>
<tr>
<td>nearly parallel</td>
<td>$\tau_0$</td>
<td>$dP = \tau_0 , Q$, $dQ = 0$</td>
</tr>
<tr>
<td>cocalibrated/semi-p.</td>
<td>$\tau_0 \oplus \tau_3$</td>
<td>$dP = \tau_0 , Q + *_7 \tau_3$, $dQ = 0$</td>
</tr>
</tbody>
</table>

8d $\text{Spin}(7)$-structures:

- $Z(G/H)$ inherits $\text{Spin}(7)$-str. def. by **self-dual 4-form $\Psi$**
  \[\Psi = P \wedge d\tau - Q\]
- $\text{Spin}(7)$-structures distinguished by **2 torsion classes**
- **Dictionary**: 7d $G_2$-structures $\leftrightarrow$ $\text{Spin}(7)$-structures on cyl. e.g. 7d loc. conf. $G_2$-str. $\rightarrow$ 8d loc. conf. $\text{Spin}(7)$-str. on cyl.
7d $G_2$-structures:
- $G_2$-str. def. by 3-form $P$ (Hodge dual 4-form $Q := \ast_7 P$)
- $G_2$-structures distinguished/classified by 4 torsion classes:
  \[
  dP = \tau_0 Q + 3 \tau_1 \wedge P + \ast_7 \tau_3 , \quad dQ = 4 \tau_1 \wedge Q + \tau_2 \wedge P
  \]
- Important examples:
  \[
  \begin{array}{|c|c|l|}
  \hline
  \text{Type} & \text{TCs} & \text{Properties} \\
  \hline
  \text{parallel} & \emptyset & dP = 0, \ dQ = 0 \\
  \text{nearly parallel} & \tau_0 & dP = \tau_0 Q, \ dQ = 0 \\
  \text{cocalibrated/semi-p.} & \tau_0 \oplus \tau_3 & dP = \tau_0 Q + \ast_7 \tau_3 , \ dQ = 0 \\
  \hline
  \end{array}
  \]

8d Spin(7)-structures:
- $Z(G/H)$ inherits Spin(7)-str. def. by self-dual 4-form $\Psi$
  \[
  \Psi = P \wedge d\tau - Q
  \]
- Spin(7)-structures distinguished by 2 torsion classes
- Dictionary: 7d $G_2$-structures $\leftrightarrow$ Spin(7)-structures on cyl.
  e.g. 7d loc. conf. $G_2$-str. $\rightarrow$ 8d loc. conf. Spin(7)-str. on cyl.
7d $G_2$-structures:

- $G_2$-str. def. by 3-form $P$ (Hodge dual 4-form $Q := \ast_7 P$)
- $G_2$-structures distinguished/classified by 4 torsion classes:

$$dP = \tau_0 \ Q + 3 \tau_1 \wedge P + \ast_7 \tau_3 \ , \quad dQ = 4 \tau_1 \wedge Q + \tau_2 \wedge P$$

- Important examples:

<table>
<thead>
<tr>
<th>Type</th>
<th>TCs</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallel</td>
<td>$\emptyset$</td>
<td>$dP = 0, dQ = 0$</td>
</tr>
<tr>
<td>nearly parallel</td>
<td>$\tau_0$</td>
<td>$dP = \tau_0 \ Q, dQ = 0$</td>
</tr>
<tr>
<td>cocalibrated/semi-p.</td>
<td>$\tau_0 \oplus \tau_3$</td>
<td>$dP = \tau_0 \ Q + \ast_7 \tau_3, dQ = 0$</td>
</tr>
</tbody>
</table>

8d Spin(7)-structures:

- $Z(G/H)$ inherits Spin(7)-str. def. by self-dual 4-form $\Psi$

$$\Psi = P \wedge d\tau - Q$$

- Spin(7)-structures distinguished by 2 torsion classes

- **Dictionary**: 7d $G_2$-structures $\leftrightarrow$ Spin(7)-structures on cyl.
  e.g. 7d loc. conf. $G_2$-str. $\rightarrow$ 8d loc. conf. Spin(7)-str. on cyl.
Back to YM theory on $Z(G/H)$

“Natural” $G$-invariant ansatz on $Z(G/H)$:

$$A = e^i l_i + e^a X_a(\tau)$$

(temporal gauge: no $d\tau$ term)

[Bauer, Ivanova, Lechtenfeld, Lubbe (2010); ...]

Notation:

- Lie algebra decomposes: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ \quad ($\mathfrak{m} \leftrightarrow G/H$)
- Lie algebra generators of $\mathfrak{g}$ split: $\{I_A\} = \{l_i\} \cup \{l_a\}$
- Lie algebra:

$$[l_i, l_j] = f_{ij}^k l_k, \quad [l_i, l_a] = f_{ia}^b l_b, \quad [l_a, l_b] = f_{ab}^i l_i + f_{ab}^c l_c$$

- $X_a(\tau) \in \mathfrak{g}$ and $\{e^i = e^i_a e^a\}$ LI 1-forms on $G/H$ dual to $\{l_i\}$

$G$-invariance condition:

$$[l_i, X_a] = f_{ia}^b X_b$$
Back to YM theory on $Z(G/H)$

“Natural” $G$-invariant ansatz on $Z(G/H)$:
\[
A = e^i l_i + e^a X_a(\tau)
\]  
(temporal gauge: no $d\tau$ term)

[Bauer, Ivanova, Lechtenfeld, Lubbe (2010); ...]

Notation:
- Lie algebra decomposes: $g = \mathfrak{h} \oplus \mathfrak{m}$  
  ($\mathfrak{m} \leftrightarrow G/H$)
- Lie algebra generators of $g$ split: $\{l_A\} = \{l_i\} \cup \{l_a\}$
- Lie algebra:
  \[
  [l_i, l_j] = f^k_{ij} l_k, \quad [l_i, l_a] = f^b_{ia} l_b, \quad [l_a, l_b] = f_i^{ab} l_i + f_c^{ab} l_c
  \]
- $X_a(\tau) \in g$ and $\{e^i = e^i_a e^a\}$ LI 1-forms on $G/H$ dual to $\{l_i\}$

$G$-invariance condition:
\[
[l_i, X_a] = f_i^b X_b
\]
Back to YM theory on $Z(G/H)$

“Natural” $G$-invariant ansatz on $Z(G/H)$:

$$A = e^i l_i + e^a X_a(\tau)$$  (temporal gauge: no $d\tau$ term)

[Bauer, Ivanova, Lechtenfeld, Lubbe (2010); ...]

Notation:

- Lie algebra decomposes: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  ($\mathfrak{m} \leftrightarrow G/H$)
- Lie algebra generators of $\mathfrak{g}$ split: $\{l_A\} = \{l_i\} \cup \{l_a\}$
- Lie algebra:
  $$\begin{align*}
  [l_i, l_j] &= f_{ij}^k l_k, & [l_i, l_a] &= f_{ia}^b l_b, & [l_a, l_b] &= f_{ab}^i l_i + f_{ab}^c l_c
  \end{align*}$$
  $$X_a(\tau) \in \mathfrak{g} \text{ and } \{e^i = e^i_a e^a\} \text{ LI 1-forms on } G/H \text{ dual to } \{l_i\}$$

$G$-invariance condition:

$$[l_i, X_a] = f_{ia}^b X_b$$
Back to YM theory on $Z(G/H)$

“Natural” $G$-invariant ansatz on $Z(G/H)$:

$$A = e^i l_i + e^a X_a(\tau)$$

(temporal gauge: no $d\tau$ term)

[Bauer, Ivanova, Lechtenfeld, Lubbe (2010); ...]

Notation:

- Lie algebra decomposes: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (m $\leftrightarrow$ G/H)
- Lie algebra generators of $\mathfrak{g}$ split: $\{l_A\} = \{l_i\} \cup \{l_a\}$
- Lie algebra:

$$[l_i, l_j] = f_{ij}^k l_k, \quad [l_i, l_a] = f_{ia}^b l_b, \quad [l_a, l_b] = f_{ab}^i l_i + f_{ab}^c l_c$$

- $X_a(\tau) \in \mathfrak{g}$ and $\{e^i = e^i_a e^a\}$ LI 1-forms on $G/H$ dual to $\{l_i\}$

$G$-invariance condition:

$$[l_i, X_a] = f_{ia}^b X_b$$
Specialize to $\mathcal{M} = Z(G/H)$ and 7d $G/H$ having $G_2$-structure.

Consider Spin(7)-instantons ($Q_{\mathcal{M}} = \Psi = \ast \Psi$):

\[ \ast F = -F \wedge \Psi \]

Insert ansatz for $A$ (note $\dot{\cdot} := \frac{d}{d\tau}(\cdot)$):

\[ \dot{X}_a + \frac{1}{2} P_{a}^{bc} \left( f_{bc}^i l_i + f_{bc}^d X_d - [X_b, X_c] \right) = 0 \]

Can’t be solved in general (depends on choice of $f_{BC}^A$)!

Single field reduction $X_a(\tau) = \phi(\tau) l_a$ — common sol. $\forall G/H$

w/ additional assumptions on $f_{BC}^A$:

\[ \dot{\phi} = \frac{\alpha \sigma}{2} \phi (\phi - 1) \]

2 static solutions: $\phi = 0, 1$.

Interpolating tanh-kink:

\[ \phi(\tau) = \frac{1}{2} \left( 1 - \tanh \left[ \frac{\alpha \sigma}{4} (\tau - \tau_0) \right] \right) \]

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Specialize to $\mathcal{M} = Z(G/H)$ and 7d $G/H$ having $G_2$-structure

Consider Spin(7)-instantons ($Q_\mathcal{M} = \Psi = \ast \Psi$):

$$\ast F = - F \wedge \Psi$$

Insert ansatz for $A$ (note $\dot{\cdot} := \frac{d}{d\tau}(\cdot)$):

$$\dot{X}_a + \frac{1}{2} P_{a}{}^{bc} \left( f^i_{bc} l_i + f^d_{bc} X_d - [X_b, X_c] \right) = 0$$

Can’t be solved in general (depends on choice of $f^A_{BC}$!)

Single field reduction $X_a(\tau) = \phi(\tau) l_a$ — common sol. $\forall G/H$

w/ additional assumptions on $f^A_{BC}$:

$$\dot{\phi} = \frac{\alpha \sigma}{2} \phi (\phi - 1)$$

2 static solutions: $\phi = 0, 1$. Interpolating tanh-kink:

$$\phi(\tau) = \frac{1}{2} \left( 1 - \tanh \left[ \frac{\alpha \sigma}{4} (\tau - \tau_0) \right] \right)$$

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Specialize to $\mathcal{M} = Z(G/H)$ and 7d $G/H$ having $G_2$-structure

Consider \textbf{Spin(7)-instantons} ($\nabla_{\mathcal{M}} = \Psi = \star \Psi$):

$$\star F = -F \wedge \Psi$$

Insert ansatz for $A$ (note $\ddot{\cdot} := \frac{d}{d\tau} (\cdot)$):

$$\ddot{X}_a + \frac{1}{2} P^{b c}_a \left( f^i_{b c} l_i + f^d_{b c} X_d - [X_b, X_c] \right) = 0$$

Can’t be solved in general (depends on choice of $f^A_{BC}$)!

Single field reduction $X_a(\tau) = \phi(\tau) l_a$ — common sol. $\forall G/H$

w/ additional assumptions on $f^A_{BC}$:

$$\dot{\phi} = \frac{\alpha \sigma}{2} \phi (\phi - 1)$$

2 static solutions: $\phi = 0, 1$.

Interpolating tanh-kink:

$$\phi(\tau) = \frac{1}{2} (1 - \tanh \left[ \frac{\alpha \sigma}{4} (\tau - \tau_0) \right])$$

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Specialize to $\mathcal{M} = Z(G/H)$ and 7d $G/H$ having $G_2$-structure

Consider **Spin(7)-instantons** $(Q_M = \Psi = \star \Psi)$:

$$\star F = - F \wedge \Psi$$

Insert ansatz for $A$ (note $(\cdot) := \frac{d}{d\tau} (\cdot)$):

$$\dot{X}_a + \frac{1}{2} P_a^{bc} \left( f_{bc}^i I_i + f_{bc}^d X_d - [X_b, X_c] \right) = 0$$

Can’t be solved in general (depends on choice of $f_{BC}^A$)!

Single field reduction $X_a(\tau) = \phi(\tau) I_a$ — **common sol.** $\forall G/H$

w/ additional assumptions on $f_{BC}^A$:

$$\dot{\phi} = \frac{\alpha \sigma}{2} \phi (\phi - 1)$$

2 static solutions: $\phi = 0, 1$.

Interpolating tanh-kink:

$$\phi(\tau) = \frac{1}{2} \left( 1 - \tanh \left[ \frac{\alpha \sigma}{4} (\tau - \tau_0) \right] \right)$$

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Specialize to $\mathcal{M} = Z(G/H)$ and 7d $G/H$ having $G_2$-structure

Consider **Spin(7)-instantons** ($Q_\mathcal{M} = \Psi = \ast \Psi$):

$$\ast F = - F \wedge \Psi$$

Insert ansatz for $A$ (note $(\cdot) := \frac{d}{d\tau} (\cdot)$):

$$\dot{X}_a + \frac{1}{2} P_a^{bc} \left( f_{b}^{i} I_{i} + f_{bc}^{d} X_{d} - [X_b, X_c] \right) = 0$$

**Can’t be solved in general** (depends on choice of $f_{BC}^A$)!

Single field reduction $X_a(\tau) = \phi(\tau) I_a$ — **common sol.** $\forall G/H$

w/ additional assumptions on $f_{BC}^A$:

$$\dot{\phi} = \frac{\alpha \sigma}{2} \phi (\phi - 1)$$

2 static solutions: $\phi = 0, 1$.

Interpolating tanh-kink:

$$\phi(\tau) = \frac{1}{2} \left( 1 - \tanh \left[ \frac{\alpha \sigma}{4} (\tau - \tau_0) \right] \right)$$

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Specialize to $\mathcal{M} = Z(G/H)$ and 7d $G/H$ having $G_2$-structure

Consider **Spin(7)-instantons** ($Q_{\mathcal{M}} = \Psi = \ast \Psi$):

$$\ast F = - F \wedge \Psi$$

Insert ansatz for $A$ (note $(\cdot) := \frac{d}{d\tau} (\cdot)$):

$$\dot{X}_a + \frac{1}{2} P_a^{\ bc} \left( f_b^i l_i + f_{bc}^d X_d - [X_b, X_c] \right) = 0$$

**Can’t be solved in general** (depends on choice of $f_{BC}^A$!)

Single field reduction $X_a(\tau) = \phi(\tau) l_a$ — **common sol.** $\forall G/H$

w/ additional assumptions on $f_{BC}^A$:

$$\dot{\phi} = \frac{\alpha \sigma}{2} \phi(\phi - 1)$$

2 static solutions: $\phi = 0, 1$.

**Interpolating tanh-kink**:

$$\phi(\tau) = \frac{1}{2} \left( 1 - \tanh \left[ \frac{\alpha \sigma}{4} (\tau - \tau_0) \right] \right)$$

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Specialize to $\mathcal{M} = Z(G/H)$ and 7d $G/H$ having $G_2$-structure

Consider **Spin(7)-instantons** ($Q_{\mathcal{M}} = \Psi = \ast \Psi$):

$$\ast F = -F \wedge \Psi$$

Insert ansatz for $A$ (note $(\dot{\cdot}) := \frac{d}{d\tau}(\cdot))$:

$$\dot{X}_a + \frac{1}{2} P_a^{bc} \left( f^i_{bc} I_i + f^d_{bc} X_d - [X_b, X_c] \right) = 0$$

**Can’t be solved in general** (depends on choice of $f_{BC}^A$)!

Single field reduction $X_a(\tau) = \phi(\tau) I_a$ — **common sol.** $\forall G/H$

w/ additional assumptions on $f_{BC}^A$:

$$\dot{\phi} = \frac{\alpha \sigma}{2} \phi (\phi - 1)$$

2 static solutions: $\phi = 0, 1$.

**Interpolating tanh-kink**:

$$\phi(\tau) = \frac{1}{2} \left( 1 - \tanh \left[ \frac{\alpha \sigma}{4} (\tau - \tau_0) \right] \right)$$

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Other (known) universal YM-solutions:

- Now, consider **YM-eq. w/ torsion** \( D \ast F + F \wedge \ast H = 0 \)
- Insert ansatz for \( A \):
  \[
  \sum_a [X_a, \dot{X}_a] = 0
  \]
  Gauss-law constraint

\[
\ddot{X}_a = \left( \frac{1}{2}(f_{acd} - H_{acd})f_{bcd} - f_{aci}f_{bcj} \right) X_b
- \frac{1}{2}(3f_{abc} - H_{abc})[X_b, X_c] - [X_b, [X_b, X_a]] - \frac{1}{2}H_{abc}f_{ibc}l_i
\]

- Single field reduction + other assumptions (\( H \propto \kappa P, \ldots \)):
  \[
  \ddot{\phi} = \frac{1}{2}(1 + \alpha)\phi(\phi - 1) \left( \phi - \frac{(\kappa + 2)\alpha - 1}{\alpha + 1} \right)
  \]
  - Newtonian mech. of pt. particle w/ quartic potential
  - \( \alpha = 0 \quad \Rightarrow \quad \phi^4 \text{ kink/anti-kink} \quad \phi = \pm \tanh \frac{\tau - \tau_0}{2} \)
  - \( (\alpha, \kappa) = (3/5, 1) \quad \Rightarrow \quad \text{Spin}(7)\text{-instantons} \)

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Other (known) universal YM-solutions:

- Now, consider **YM-eq. w/ torsion** $D \ast F + F \wedge \ast H = 0$
- Insert ansatz for $A$:
  \[
  \sum_a [X_a, \dot{X}_a] = 0 \quad \text{Gauss-law constraint}
  \]
  \[
  \ddot{X}_a = \left( \frac{1}{2} (f_{acd} - H_{acd}) f_{bcd} - f_{aci} f_{bci} \right) X_b
  - \frac{1}{2} (3f_{abc} - H_{abc}) [X_b, X_c] - [X_b, [X_b, X_a]] - \frac{1}{2} H_{abc} f_{ibc} I_i
  \]
- Single field reduction + other assumptions ($H \propto \kappa P$, ...):
  \[
  \ddot{\phi} = \frac{1}{2} (1 + \alpha) \phi (\phi - 1) \left( \phi - \frac{(\kappa + 2)\alpha - 1}{\alpha + 1} \right)
  \]
  - Newtonian mech. of pt. particle w/ quartic potential
  - $\alpha = 0 \quad \rightarrow \quad \phi^4 \text{ kink/anti-kink} \quad \phi = \pm \tanh \frac{\tau - \tau_0}{2}$
  - $(\alpha, \kappa) = (3/5, 1) \rightarrow \text{Spin(7)-instantons}$

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Other (known) universal YM-solutions:

- Now, consider **YM-eq. w/ torsion** \( D^* F + F \wedge *H = 0 \)
- Insert ansatz for \( A \):
  \[
  \sum_a [X_a, \dot{X}_a] = 0 \tag{Gauss-law constraint}
  \]
  \[
  \ddot{X}_a = \left( \frac{1}{2} (f_{acd} - H_{acd}) f_{bcd} - f_{aci} f_{bci} \right) X_b - \frac{1}{2} (3f_{abc} - H_{abc}) [X_b, X_c] - [X_b, [X_b, X_a]] - \frac{1}{2} H_{abc} f_{ibc} l_i
  \]
- Single field reduction + other assumptions \((H \propto \kappa P, \ldots)\):
  \[
  \ddot{\phi} = \frac{1}{2} (1 + \alpha) \phi (\phi - 1) \left( \phi - \frac{(\kappa + 2) \alpha - 1}{\alpha + 1} \right)
  \]
  - Newtonian mech. of pt. particle w/ quartic potential
  - \( \alpha = 0 \rightarrow \phi^4 \) **kink/anti-kink** \( \phi = \pm \tanh \frac{\tau - \tau_0}{2} \)
  - \((\alpha, \kappa) = (3/5, 1) \rightarrow \text{Spin}(7)-instantons**

[Ivanova, Lechtenfeld, Popov, Rahn (2009)]
Case-by-case analysis:
Consider **multi-field** configurations . . .

- . . . on cylinders over **three** 7d cosets with nearly parallel $G_2$-structure
  - Berger space $SO(5)/SO(3)_{\text{max}}$
  - Squashed 7-sphere $Sp(2) \times Sp(1)/Sp(1)^2$
  - (Aloff-Wallach spaces $SU(3)/U(1)_{k,l}$, cf. also [AH, Ivanova, Lechtenfeld, Popov (2011); Geipel (2016)])

- . . . and on cylinders over **four** 7d cosets with $SU(3)$-structure ($SU(3) \subset G_2$, special case of $G_2$-struct.)
  - $(SO(5)/SO(3)_{A+B})$
  - $(N^{pqr} = (SU(3) \times U(1))/(U(1) \times U(1)))$
  - $M^{pqr} = (SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1) \times U(1))$
  - $Q^{pqr} = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$

- Present some of the **new solutions** in the following
Case-by-case analysis:
Consider **multi-field** configurations . . .

- . . . on cylinders over **three** 7d cosets with nearly parallel $G_2$-structure
  - Berger space $SO(5)/SO(3)_{\text{max}}$
  - Squashed 7-sphere $Sp(2) \times Sp(1)/Sp(1)^2$
  - (Aloff-Wallach spaces $SU(3)/U(1)_{k,l}$, cf. also [AH, Ivanova, Lechtenfeld, Popov (2011); Geipel (2016)])

- . . . and on cylinders over **four** 7d cosets with $SU(3)$-structure ($SU(3) \subset G_2$, special case of $G_2$-struct.)
  - $(SO(5)/SO(3)_{A+B})$
  - $(N^{pqr} = (SU(3) \times U(1))/(U(1) \times U(1)))$
  - $M^{pqr} = (SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1) \times U(1))$
  - $Q^{pqr} = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$

- Present some of the new solutions in the following
Case-by-case analysis:
Consider **multi-field** configurations . . .

- . . . on cylinders over **three** 7d cosets with nearly parallel $G_2$-structure
  - Berger space $SO(5)/SO(3)_{\text{max}}$
  - Squashed 7-sphere $Sp(2) \times Sp(1)/Sp(1)^2$
  - (Aloff-Wallach spaces $SU(3)/U(1)_{k,l}$, cf. also [AH, Ivanova, Lechtenfeld, Popov (2011); Geipel (2016)])

- . . . and on cylinders over **four** 7d cosets with $SU(3)$-structure ($SU(3) \subset G_2$, special case of $G_2$-struct.)
  - $(SO(5)/SO(3)_{A+B})$
  - $(N^{pqr} = (SU(3) \times U(1))/(U(1) \times U(1)))$
  - $M^{pqr} = (SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1) \times U(1))$
  - $Q^{pqr} = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$

- Present some of the **new solutions** in the following
Berger space & squashed $S^7$:

- 1st step to determine multi-field sol:
  solve $G$-inv. cond. $[l_i, X_a] = f_{ia}^b X_b$

- Berger space: $G$-inv. cond. $\Rightarrow X_a = \phi l_a$
  (back to single field case: nothing new)

- Squashed $S^7$: $G$-inv. cond. $\Rightarrow$ 2 real fields $\phi_1$, $\phi_2$
  - Instanton eq.: $\phi_1 = \pm \phi_2 \equiv \pm \phi$ (again, back to old case)
  - YM-eq. $\Rightarrow$ 2 branches:
    - “instanton branch” ($\phi_1 = \pm \phi_2 \equiv \pm \phi$)
      $\rightarrow$ single-field case (Spin(7)-instantons + $\phi^4$ (anti-)kink)
    - $\phi_2 = (\kappa + 3)/2$, $\phi_1(\tau) = \pm \sqrt{\kappa c} \tanh \left[ \frac{\sqrt{\kappa c}}{2} (\tau - \tau_0) \right]$
      flat direction + single rescaled $\phi^4$ (anti-)kink
Berger space & squashed $S^7$:

- 1st step to determine multi-field sol: solve $G$-inv. cond. $[I_i, X_a] = f_{ia}^b X_b$
- **Berger space**: $G$-inv. cond. $\implies X_a = \phi l_a$
  (back to single field case: **nothing new**)
- **Squashed $S^7$**: $G$-inv. cond. $\implies$ 2 real fields $\phi_1, \phi_2$
  - Instanton eq.: $\phi_1 = \pm \phi_2 \equiv \pm \phi$ (again, back to old case)
  - YM-eq. $\implies$ 2 branches:
    - "instanton branch" ($\phi_1 = \pm \phi_2 \equiv \pm \phi$
      $\rightarrow$ single-field case (Spin(7)-instantons + $\phi^4$ (anti-)kink)
    - $\phi_2 = (\kappa + 3)/2$, $\phi_1(\tau) = \pm \sqrt{c\kappa} \tanh \left[ \frac{\sqrt{c\kappa}}{2} (\tau - \tau_0) \right]$
      flat direction + single rescaled $\phi^4$ (anti-)kink
Berger space & squashed $S^7$: 

- 1st step to determine multi-field sol: 
  solve *G-inv. cond.* $[l_i, X_a] = f_{ia}^b X_b$

- **Berger space**: *G-inv. cond.* $\Rightarrow X_a = \phi l_a$
  (back to single field case: **nothing new**)

- **Squashed $S^7$**: *G-inv. cond.* $\Rightarrow$ 2 real fields $\phi_1, \phi_2$
  
  - Instanton eq.: $\phi_1 = \pm \phi_2 \equiv \mp \phi$ (again, **back to old case**)
  
  - YM-eq. $\Rightarrow$ 2 branches:
    
    1. “instanton branch” ($\phi_1 = \pm \phi_2 \equiv \pm \phi$)
       $\rightarrow$ single-field case (Spin(7)-instantons + $\phi^4$ (anti-)kink)
    
    2. $\phi_2 = (\kappa + 3)/2, \phi_1(\tau) = \pm \sqrt{c_\kappa} \tanh \left[ \frac{\sqrt{c_\kappa}}{2} (\tau - \tau_0) \right]$

      flat direction + single rescaled $\phi^4$ (anti-)kink
Berger space & squashed $S^7$:

- 1st step to determine multi-field sol:
  solve $G$-inv. cond. $[l_i, X_a] = f^b_{ia} X_b$

- **Berger space**: $G$-inv. cond. $\implies X_a = \phi l_a$
  (back to single field case: **nothing new**)

- **Squashed $S^7$**: $G$-inv. cond. $\implies 2$ real fields $\phi_1, \phi_2$
  
  - Instanton eq.: $\phi_1 = \pm \phi_2 \equiv \pm \phi$ (again, **back to old case**)
  
  - YM-eq. $\implies 2$ branches:
    
    1. “instanton branch” ($\phi_1 = \pm \phi_2 \equiv \pm \phi$)
       $\to$ single-field case (Spin(7)-instantons + $\phi^4$ (anti-)kink)
    
    2. $\phi_2 = (\kappa + 3)/2$, $\phi_1(\tau) = \pm \sqrt{c_\kappa} \tanh \left[ \frac{\sqrt{c_\kappa}}{2} (\tau - \tau_0) \right]$
       flat direction + single rescaled $\phi^4$ (anti-)kink
Berger space & squashed $S^7$:

- 1st step to determine multi-field sol:
  solve $G$-inv. cond. $[l_i, X_a] = f^b_{ia} X_b$
- **Berger space**: $G$-inv. cond. $\implies X_a = \phi l_a$
  (back to single field case: **nothing new**)
- **Squashed $S^7$**: $G$-inv. cond. $\implies$ 2 real fields $\phi_1, \phi_2$
  - Instanton eq.: $\phi_1 = \pm \phi_2 \equiv \pm \phi$ (again, back to old case)
  - YM-eq. $\implies$ 2 branches:
    1. "instanton branch" ($\phi_1 = \pm \phi_2 \equiv \pm \phi$)
      $\rightarrow$ single-field case (Spin(7)-instantons + $\phi^4$ (anti-)kink)
    2. $\phi_2 = (\kappa + 3)/2$, $\phi_1(\tau) = \pm \sqrt{c_\kappa} \tanh \left[ \frac{\sqrt{c_\kappa}}{2} (\tau - \tau_0) \right]$
      flat direction + single rescaled $\phi^4$ (anti-)kink
Berger space & squashed $S^7$:

- 1st step to determine multi-field sol:
  solve $G$-inv. cond. $[l_i, X_a] = f_{ia}^b X_b$

- **Berger space**: $G$-inv. cond. $\implies X_a = \phi l_a$
  (back to single field case: **nothing new**)

- **Squashed $S^7$**: $G$-inv. cond. $\implies$ 2 real fields $\phi_1, \phi_2$
  - Instanton eq.: $\phi_1 = \pm \phi_2 \equiv \pm \phi$ (again, back to old case)
  - YM-eq. $\implies$ 2 branches:

  1. "**instanton branch**" ($\phi_1 = \pm \phi_2 \equiv \pm \phi$)
     $\rightarrow$ single-field case (Spin(7)-instantons + $\phi^4$ (anti-)kink)

  2. $\phi_2 = (\kappa + 3)/2$, $\phi_1(\tau) = \pm \sqrt{c_\kappa} \tanh \left[ \frac{\sqrt{c_\kappa}}{2} (\tau - \tau_0) \right]$
     flat direction + single rescaled $\phi^4$ (anti-)kink
Berger space & squashed $S^7$:

- 1st step to determine multi-field sol:
  solve $G$-inv. cond. $[l_i, X_a] = f_{ia}^b X_b$

- **Berger space**: $G$-inv. cond. $\implies X_a = \phi l_a$
  (back to single field case: **nothing new**)

- **Squashed $S^7$**: $G$-inv. cond. $\implies 2$ real fields $\phi_1, \phi_2$
  - Instanton eq.: $\phi_1 = \pm \phi_2 \equiv \pm \phi$ (again, **back to old case**)
  - YM-eq. $\implies 2$ branches:
    1. **“instanton branch”** ($\phi_1 = \pm \phi_2 \equiv \pm \phi$)
       $\rightarrow$ single-field case (Spin(7)-instantons $+$ $\phi^4$ (anti-)kink)
    2. $\phi_2 = (\kappa + 3)/2$, $\phi_1(\tau) = \pm \sqrt{c_\kappa} \tanh \left[ \frac{\sqrt{c_\kappa}}{2} (\tau - \tau_0) \right]$
       flat direction + single rescaled $\phi^4$ (anti-)kink
Non-trivial multi-field solution I ([AH (2016)]):

**1st example:** \( Z(M^{pqr}) \), \( M^{pqr} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)} \)

- **SU(3)-structure only** for \( p = \pm q, r = 0 \). W.l.o.g. take \( M^{110} \)
- **G-inv. cond.** \( \implies \) 5 real fields \( \phi_1, ..., \phi_5 \)
- Gauss-law constraint \( \implies \phi_4 \sim \phi_1, \phi_5 \sim \phi_2 \)
- Analytical sector \( \phi_3 = 1/(2\sqrt{2}) \) (+ fixing of other parameters).

Remaining 2d motion:
Non-trivial multi-field solution I ([AH (2016)]):

- **1st example:** $Z(M^{pq})$, $M^{pq} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)}$
  - $SU(3)$-structure only for $p = \pm q$, $r = 0$. W.l.o.g. take $M^{110}$
  - $G$-inv. cond. $\implies$ 5 real fields $\phi_1, \ldots, \phi_5$
  - Gauss-law constraint $\implies$ $\phi_4 \sim \phi_1$, $\phi_5 \sim \phi_2$
  - Analytical sector $\phi_3 = 1/(2\sqrt{2})$ (+ fixing of other parameters).

  Remaining 2d motion:
Non-trivial multi-field solution I ([AH (2016)]):

- **1st example:** \( Z(M^{pqr}) \), \( M^{pqr} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)} \)
  - \( SU(3) \)-structure **only** for \( p = \pm q, r = 0 \). W.l.o.g. take \( M^{110} \)
  - \( G \)-inv. cond. \( \implies \) 5 real fields \( \phi_1, ..., \phi_5 \)
  - Gauss-law constraint \( \implies \phi_4 \sim \phi_1, \phi_5 \sim \phi_2 \)
  - Analytical sector \( \phi_3 = 1/(2\sqrt{2}) \) (+ fixing of other parameters).

Remaining 2d motion:
Non-trivial multi-field solution I ([AH (2016)]):

1st example: \( Z(M^{pqr} \), \( M^{pqr} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)} \)

- \( SU(3) \)-structure only for \( p = \pm q \), \( r = 0 \). W.l.o.g. take \( M^{110} \)
- \( G \)-inv. cond. \( \implies 5 \) real fields \( \phi_1, ..., \phi_5 \)
- Gauss-law constraint \( \implies \phi_4 \sim \phi_1, \phi_5 \sim \phi_2 \)
- Analytical sector \( \phi_3 = 1/(2\sqrt{2}) \) (+ fixing of other parameters).

Remaining 2d motion:
Non-trivial multi-field solution I ([AH (2016)]):

- **1st example**: $Z(M^{pq r})$, $M^{pq r} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)}$
  - $SU(3)$-structure **only** for $p = \pm q$, $r = 0$. W.l.o.g. take $M^{110}$
  - $G$-inv. cond. $\implies$ 5 real fields $\phi_1$, ..., $\phi_5$
  - Gauss-law constraint $\implies$ $\phi_4 \sim \phi_1$, $\phi_5 \sim \phi_2$
  - Analytical sector $\phi_3 = 1/(2\sqrt{2})$ (+ fixing of other parameters).

Remaining 2d motion:

**Analytical multi-field solutions** (of YM w/ torsion)
- **Blue**: finite-energy (physical) YM-configs. **Green**: $E \to \infty$. 
Non-trivial multi-field solution II ([AH (2016))):

- **2nd example**: $Z(Q^{pqr})$, $Q^{pqr} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}$

  - $SU(3)$-structure **only** for $p = \pm q, q = \pm r$. W.l.o.g. take $Q^{111}$
  - $G$-inv. cond. $\implies$ 7 real fields $\phi_1, ..., \phi_7$
  - Gauss-law constraint $\implies \phi_5 \sim \phi_1, \phi_6 \sim \phi_2, \phi_7 \sim \phi_3$
  - and $\phi_1^2 = \phi_2^2 = \phi_3^2$ (uninteresting) or $\phi_4 = (2\lambda + 3)/(2\sqrt{2})$
  - Remaining dynamics in $\phi_1, \phi_2, \phi_3$ **decouples**, e.g.
    \[
    \mathcal{L} = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \dot{\phi}_\alpha^2 + \frac{1}{8} \left( \phi_\alpha^2 - (c_7^\pm)^2 \right)^2 \right\}, \quad c_7^\pm := \sqrt{9 \pm 2\sqrt{15}}
    \]

  - 3-vector of independent **rescaled** $\phi^4$ kinks-/anti-kinks
    \[
    \phi = c_7^\pm \begin{pmatrix}
    \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,1}) \right] \\
    \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,2}) \right] \\
    \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,3}) \right]
    \end{pmatrix}
    \]

  - Interpolates between $(\pm c_7^\pm, \pm c_7^\pm, \pm c_7^\pm)$ as $\tau \to \pm \infty$
  - Finite energy (physically allowed)
Non-trivial multi-field solution II ([AH (2016)]):

- **2nd example**: $Z(Q^{pqr})$, $Q^{pqr} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}$

- $SU(3)$-structure **only** for $p = \pm q$, $q = \pm r$. W.l.o.g. take $Q^{111}$

- $G$-inv. cond. $\implies$ 7 real fields $\phi_1, \ldots, \phi_7$

- Gauss-law constraint $\implies \phi_5 \sim \phi_1$, $\phi_6 \sim \phi_2$, $\phi_7 \sim \phi_3$

- and $\phi_1^2 = \phi_2^2 = \phi_3^2$ (uninteresting) or $\phi_4 = (2\lambda + 3)/(2\sqrt{2})$

- Remaining dynamics in $\phi_1$, $\phi_2$, $\phi_3$ **decouples**, e.g.

$$L = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \phi_\alpha^2 + \frac{1}{8} \left( \phi_\alpha^2 - (c_7^\pm)^2 \right)^2 \right\}, \quad c_7^\pm := \sqrt{9 \pm 2\sqrt{15}}$$

- 3-vector of independent **rescaled** $\phi^4$ kinks-/anti-kinks

$$\phi = c_7^\pm \begin{pmatrix} \pm \tanh \left( \frac{c_7^\pm}{2} (\tau - \tau_{0,1}) \right) \\ \pm \tanh \left( \frac{c_7^\pm}{2} (\tau - \tau_{0,2}) \right) \\ \pm \tanh \left( \frac{c_7^\pm}{2} (\tau - \tau_{0,3}) \right) \end{pmatrix}$$

- Interpolates between $(\pm c_7^\pm, \pm c_7^\pm, \pm c_7^\pm)$ as $\tau \to \pm \infty$

- Finite energy (physically allowed)
Non-trivial multi-field solution II ([AH (2016)]):

- **2nd example**: $Z(Q^{pqr})$, $Q^{pqr} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}$

- $SU(3)$-structure **only** for $p = \pm q$, $q = \pm r$. W.l.o.g. take $Q^{111}$

- $G$-inv. cond. $\implies$ 7 real fields $\phi_1, \ldots, \phi_7$

- Gauss-law constraint $\implies \phi_5 \sim \phi_1, \phi_6 \sim \phi_2, \phi_7 \sim \phi_3$

- and $\phi_1^2 = \phi_2^2 = \phi_3^2$ (uninteresting) or $\phi_4 = (2\lambda + 3)/(2\sqrt{2})$

- Remaining dynamics in $\phi_1, \phi_2, \phi_3$ **decouples**, e.g.

  $$\mathcal{L} = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \dot{\phi}_\alpha^2 + \frac{1}{8} \left( \phi_\alpha^2 - (c_{7}^{\pm})^2 \right)^2 \right\}, \quad c_{7}^{\pm} := \sqrt{9 \pm 2\sqrt{15}}$$

- 3-vector of independent **rescaled** $\phi^4$ kinks-/anti-kinks

  $$\phi = c_{7}^{\pm} \begin{pmatrix} \pm \tanh \left[ \frac{c_{7}^{\pm}}{2} (\tau - \tau_{0,1}) \right] \\ \pm \tanh \left[ \frac{c_{7}^{\pm}}{2} (\tau - \tau_{0,2}) \right] \\ \pm \tanh \left[ \frac{c_{7}^{\pm}}{2} (\tau - \tau_{0,3}) \right] \end{pmatrix}$$

- Interpolates between $(\pm c_{7}^{\pm}, \pm c_{7}^{\pm}, \pm c_{7}^{\pm})$ as $\tau \to \pm \infty$

- Finite energy (physically allowed)
Non-trivial multi-field solution II ([AH (2016)]):

- **2nd example:** $Z(Q^{pqr})$, $Q^{pqr} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}$
  
  - $SU(3)$-structure **only** for $p = \pm q$, $q = \pm r$. W.l.o.g. take $Q^{111}$
  
  - $G$-inv. cond. $\implies$ 7 real fields $\phi_1, ..., \phi_7$
  
  - Gauss-law constraint $\implies \phi_5 \sim \phi_1$, $\phi_6 \sim \phi_2$, $\phi_7 \sim \phi_3$
  
  - and $\phi_1^2 = \phi_2^2 = \phi_3^2$ (uninteresting) or $\phi_4 = (2\lambda + 3)/(2\sqrt{2})$
  
  - Remaining dynamics in $\phi_1, \phi_2, \phi_3$ **decouples**, e.g.

$$L = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \dot{\phi}_\alpha^2 + \frac{1}{8} \left( \phi_\alpha^2 - (c_7^\pm)^2 \right)^2 \right\}, \quad c_7^\pm := \sqrt{9 \pm 2\sqrt{15}}$$

- 3-vector of independent **rescaled** $\phi^4$ kinks-/anti-kinks

$$\phi = c_7^\pm \begin{pmatrix} \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,1}) \right] \\ \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,2}) \right] \\ \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,3}) \right] \end{pmatrix}$$

- Interpolates between $(\pm c_7^\pm, \pm c_7^\pm, \pm c_7^\pm)$ as $\tau \to \pm \infty$

- Finite energy (physically allowed)
Non-trivial multi-field solution II ([AH (2016))]:

- **2nd example:** $Z(Q^{pq})$, $Q^{pq} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}$

- **$SU(3)$-structure only** for $p = \pm q$, $q = \pm r$. W.l.o.g. take $Q^{111}$

- **G-inv. cond.** $\implies$ 7 real fields $\phi_1, \ldots, \phi_7$

- **Gauss-law constraint** $\implies$ $\phi_5 \sim \phi_1$, $\phi_6 \sim \phi_2$, $\phi_7 \sim \phi_3$

- and $\phi_1^2 = \phi_2^2 = \phi_3^2$ (uninteresting) or $\phi_4 = (2\lambda + 3)/(2\sqrt{2})$

- Remaining dynamics in $\phi_1, \phi_2, \phi_3$ **decouples**, e.g.

$$\mathcal{L} = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \dot{\phi}_\alpha^2 + \frac{1}{8} (\phi_\alpha^2 - (c_7^\pm)^2)^2 \right\}, \quad c_7^\pm := \sqrt{9 \pm 2\sqrt{15}}$$

- 3-vector of independent **rescaled** $\phi^4$ kinks-/anti-kinks

$$\phi = c_7^\pm \begin{pmatrix} \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,1}) \right] \\ \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,2}) \right] \\ \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,3}) \right] \end{pmatrix}$$

- Interpolates between $(\pm c_7^\pm, \pm c_7^\pm, \pm c_7^\pm)$ as $\tau \to \pm \infty$

- Finite energy (physically allowed)
Non-trivial multi-field solution II ([AH (2016)]):

- **2nd example**: \( Z(Q^{pq r}) \), \( Q^{pq r} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)} \)

- **\( SU(3) \)-structure only** for \( p = \pm q \), \( q = \pm r \). W.l.o.g. take \( Q^{111} \)

- **G-inv. cond.** \( \implies \) 7 real fields \( \phi_1, ..., \phi_7 \)

- **Gauss-law constraint** \( \implies \phi_5 \sim \phi_1, \phi_6 \sim \phi_2, \phi_7 \sim \phi_3 \)

- and \( \phi_1^2 = \phi_2^2 = \phi_3^2 \) (uninteresting) or \( \phi_4 = (2\lambda + 3)/(2\sqrt{2}) \)

- Remaining dynamics in \( \phi_1, \phi_2, \phi_3 \) **decouples**, e.g.

\[
\mathcal{L} = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \phi_\alpha^2 + \frac{1}{8} \left( \phi_\alpha^2 - (c_7^\pm)^2 \right)^2 \right\}, \quad c_7^\pm := \sqrt{9 \pm 2\sqrt{15}}
\]

- 3-vector of independent **rescaled** \( \phi^4 \) kinks-/anti-kinks

\[
\phi = c_7^\pm \begin{pmatrix}
\pm \tanh \left[ \frac{c_7^\pm}{2}(\tau - \tau_{0,1}) \right] \\
\pm \tanh \left[ \frac{c_7^\pm}{2}(\tau - \tau_{0,2}) \right] \\
\pm \tanh \left[ \frac{c_7^\pm}{2}(\tau - \tau_{0,3}) \right]
\end{pmatrix}
\]

- Interpolates between \((\pm c_7^\pm, \pm c_7^\pm, \pm c_7^\pm)\) as \( \tau \to \pm \infty \)

- Finite energy (physically allowed)
Non-trivial multi-field solution II ([AH (2016)):

- **2nd example**: $Z(Q^{pqr})$, $Q^{pqr} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}$
- $SU(3)$-structure only for $p = \pm q$, $q = \pm r$. W.l.o.g. take $Q^{111}$
- $G$-inv. cond. $\implies$ 7 real fields $\phi_1, \ldots, \phi_7$
- Gauss-law constraint $\implies$ $\phi_5 \sim \phi_1$, $\phi_6 \sim \phi_2$, $\phi_7 \sim \phi_3$
- and $\phi_1^2 = \phi_2^2 = \phi_3^2$ (uninteresting) or $\phi_4 = (2\lambda + 3)/(2\sqrt{2})$
- Remaining dynamics in $\phi_1, \phi_2, \phi_3$ decouples, e.g.

$$\mathcal{L} = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \dot{\phi}_\alpha^2 + \frac{1}{8} (\phi_\alpha^2 - (c_7^{\pm})^2)^2 \right\}, \quad c_7^{\pm} := \sqrt{9 \pm 2\sqrt{15}}$$

- 3-vector of independent *rescaled* $\phi^4$ kinks-/anti-kinks

$$\phi = c_7^{\pm} \begin{pmatrix} \pm \tanh \left[ \frac{c_7^{\pm}}{2} (\tau - \tau_{0,1}) \right] \\ \pm \tanh \left[ \frac{c_7^{\pm}}{2} (\tau - \tau_{0,2}) \right] \\ \pm \tanh \left[ \frac{c_7^{\pm}}{2} (\tau - \tau_{0,3}) \right] \end{pmatrix}$$

- Interpolates between $(\pm c_7^{\pm}, \pm c_7^{\pm}, \pm c_7^{\pm})$ as $\tau \to \pm \infty$
- Finite energy (physically allowed)
Non-trivial multi-field solution II ([AH (2016)]):

- **2nd example:** $Z(\mathcal{Q}_{pqr})$, $Q_{pqr} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}$
  
  - $SU(3)$-structure **only** for $p = \pm q$, $q = \pm r$. W.l.o.g. take $Q_{111}^{111}$
  
  - $G$-inv. cond. $\implies$ 7 real fields $\phi_1$, ..., $\phi_7$
  
  - Gauss-law constraint $\implies \phi_5 \sim \phi_1$, $\phi_6 \sim \phi_2$, $\phi_7 \sim \phi_3$
  
  - and $\phi_1^2 = \phi_2^2 = \phi_3^2$ (uninteresting) or $\phi_4 = (2\lambda + 3)/(2\sqrt{2})$
  
  - Remaining dynamics in $\phi_1$, $\phi_2$, $\phi_3$ **decouples**, e.g.

$$L = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \dot{\phi}_\alpha^2 + \frac{1}{8} \left( \phi_\alpha^2 - (c_7^\pm)^2 \right)^2 \right\}, \quad c_7^\pm := \sqrt{9 \pm 2\sqrt{15}}$$

- 3-vector of independent **rescaled** $\phi^4$ kinks-/anti-kinks

$$\phi = c_7^\pm \begin{pmatrix} \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,1}) \right] \\ \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,2}) \right] \\ \pm \tanh \left[ \frac{c_7^\pm}{2} (\tau - \tau_{0,3}) \right] \end{pmatrix}$$

- Interpolates between $(\pm c_7^\pm, \pm c_7^\pm, \pm c_7^\pm)$ as $\tau \to \pm \infty$

- Finite energy (physically allowed)
Non-trivial multi-field solution II ([AH (2016)]):

- **2nd example**: $Z(Q^{pqr})$, $Q^{pqr} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}$
  
  - **$SU(3)$-structure only** for $p = \pm q$, $q = \pm r$. W.l.o.g. take $Q^{111}$
  
  - **G-inv. cond.** $\Rightarrow$ 7 real fields $\phi_1, \ldots, \phi_7$
  
  - **Gauss-law constraint** $\Rightarrow \phi_5 \sim \phi_1, \phi_6 \sim \phi_2, \phi_7 \sim \phi_3$
  
  - and $\phi_1^2 = \phi_2^2 = \phi_3^2$ (uninteresting) or $\phi_4 = (2\lambda + 3)/(2\sqrt{2})$
  
  - Remaining dynamics in $\phi_1, \phi_2, \phi_3$ **decouples**, e.g.

  $$L = \sum_{\alpha=1}^{3} \left\{ \frac{1}{2} \dot{\phi}_{\alpha}^2 + \frac{1}{8} \left( \phi_{\alpha}^2 - (c_7^{\pm})^2 \right)^2 \right\}, \quad c_7^{\pm} := \sqrt{9 \pm 2\sqrt{15}}$$

  - 3-vector of independent **rescaled** $\phi^4$ kinks-/anti-kinks

    $$\phi = c_7^{\pm} \begin{pmatrix} \pm \tanh \left[ \frac{c_7^{\pm}}{2} (\tau - \tau_{0,1}) \right] \\ \pm \tanh \left[ \frac{c_7^{\pm}}{2} (\tau - \tau_{0,2}) \right] \\ \pm \tanh \left[ \frac{c_7^{\pm}}{2} (\tau - \tau_{0,3}) \right] \end{pmatrix}$$

  - Interpolates between $(\pm c_7^{\pm}, \pm c_7^{\pm}, \pm c_7^{\pm})$ as $\tau \to \pm \infty$
  
  - **Finite energy** (physically allowed)
Summary

1. Higher-dim. **YM instantons** obey $*F = -F \wedge *Q_M$
2. Higher-dim. **YM theory w/ torsion**: $D *F + F \wedge *H = 0$
3. Both arise naturally in S.T. together with $G$-structure
4. Studied on $Z(G/H) = \mathbb{R} \times G/H$. $G/H$: 7d, $G_2/SU(3)$-str.:
   - (1) reduces to gradient flow eqs
   - (2) reduces to Newtonian mechanics of pt. particle moving in $\mathbb{R}^n$ w/ quartic potential (+ constraints)
   - found plethora of new numerical & analytical solutions

Open Problems & WIP

- Other cosets, ansätze, corners of param./field space, ...
- Find explicit S.T. embeddings. Promising candidate: het SUGRA w/ $\mathbb{R}^{1,1} \times \mathbb{R} \times G/H +$ domain wall structure (?) (analog of [AH, Lechtenfeld, Musaev (2014)])
Summary

1. Higher-dim. **YM instantons** obey $\ast F = -F \wedge \ast Q_M$

2. Higher-dim. **YM theory w/ torsion**: $D \ast F + F \wedge \ast H = 0$

3. Both arise naturally in S.T. together with $G$-structure

4. Studied on $Z(G/H) = \mathbb{R} \times G/H$. $G/H$: 7d, $G_2/SU(3)$-str.:
   - (1) reduces to gradient flow eqs
   - (2) reduces to Newtonian mechanics of pt. particle moving in $\mathbb{R}^n$ w/ quartic potential (+ constraints)
   - found plethora of new numerical & analytical solutions

Open Problems & WIP

- Other cosets, ansätze, corners of param./field space, ...
- Find explicit S.T. embeddings. Promising candidate: het SUGRA w/ $\mathbb{R}^{1,1} \times \mathbb{R} \times G/H +$ domain wall structure (?) (analog of [AH, Lechtenfeld, Musaev (2014)])
**Summary**

1. **Higher-dim. YM instantons** obey $\ast F = -F \wedge \ast Q_M$
2. **Higher-dim. YM theory w/ torsion:** $D \ast F + F \wedge \ast H = 0$
3. **Both arise naturally** in S.T. together with $G$-structure
4. Studied on $Z(G/H) = \mathbb{R} \times G/H$. $G/H$: 7d, $G_2/SU(3)$-str.:  
   - (1) reduces to gradient flow eqs  
   - (2) reduces to Newtonian mechanics of pt. particle moving in $\mathbb{R}^n$ w/ quartic potential (+ constraints)  
   - found plethora of new numerical & analytical solutions

**Open Problems & WIP**

- Other cosets, ansätze, corners of param./field space, ...
- Find explicit S.T. embeddings. Promising candidate: het SUGRA w/ $\mathbb{R}^{1,1} \times \mathbb{R} \times G/H +$ domain wall structure (?)  
  (analog of [AH, Lechtenfeld, Musaev (2014)])
Summary

1. Higher-dim. **YM instantons** obey $\ast F = - F \wedge \ast Q_M$
2. Higher-dim. **YM theory w/ torsion**: $D \ast F + F \wedge \ast H = 0$
3. **Both arise naturally** in S.T. together with **G-structure**
4. Studied on $Z(G/H) = \mathbb{R} \times G/H$. $G/H$: 7d, $G_2/SU(3)$-str.:
   - (1) reduces to **gradient flow eqs**
   - (2) reduces to **Newtonian mechanics of pt. particle** moving in $\mathbb{R}^n$ w/ quartic potential (+ constraints)
   - found plethora of new numerical & analytical solutions

Open Problems & WIP

- Other cosets, ansätze, corners of param./field space, ...
- Find explicit S.T. embeddings. Promising candidate: het SUGRA w/ $\mathbb{R}^{1,1} \times \mathbb{R} \times G/H + \text{domain wall structure (?) (analog of [AH, Lechtenfeld, Musaev (2014)])}$
Summary

1. Higher-dim. **YM instantons** obey $\ast F = - F \wedge \ast Q_M$
2. Higher-dim. **YM theory w/ torsion**: $D \ast F + F \wedge \ast H = 0$
3. **Both arise naturally** in S.T. together with **G-structure**
4. Studied on $Z(G/H) = \mathbb{R} \times G/H$. $G/H$: 7d, $G_2/SU(3)$-str.:
   - (1) reduces to **gradient flow eqs**
   - (2) reduces to **Newtonian mechanics of pt. particle** moving in $\mathbb{R}^n$ w/ quartic potential (+ constraints)
   - found plethora of new numerical & analytical solutions

Open Problems & WIP

- Other cosets, ansätze, corners of param./field space, ...
- Find explicit **S.T. embeddings**. Promising candidate: het SUGRA w/ $\mathbb{R}^{1,1} \times \mathbb{R} \times G/H +$ domain wall structure (?) (analog of [AH, Lechtenfeld, Musaev (2014)])
Summary

1. Higher-dim. YM instantons obey \( \ast F = -F \wedge \ast Q_M \)
2. Higher-dim. YM theory w/ torsion: \( D \ast F + F \wedge \ast H = 0 \)
3. Both arise naturally in S.T. together with G-structure
4. Studied on \( Z(G/H) = \mathbb{R} \times G/H \). \( G/H \): 7d, \( G_2/SU(3) \)-str.:
   - (1) reduces to gradient flow eqs
   - (2) reduces to Newtonian mechanics of pt. particle moving in \( \mathbb{R}^n \) w/ quartic potential (+ constraints)
   - found plethora of new numerical & analytical solutions

Open Problems & WIP

- Other cosets, ansätze, corners of param./field space, ...
- Find explicit S.T. embeddings. Promising candidate: het SUGRA w/ \( \mathbb{R}^{1,1} \times \mathbb{R} \times G/H + \text{domain wall} \) structure (?) (analog of \([AH, Lechtenfeld, Musaev (2014)]\))
Thank you for your attention.