Universität Hamburg
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DESY

Particles, Strings, and the Early Universe
Collaborative Research Center SFB 676

Spin(7)-instantons & other Yang-Mills solutions on cylinders over coset spaces with G_2-structure

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Outline

1. Introduction
   - Motivation
   - Yang-Mills instantons in $d = 4$
   - Instantons in $d > 4$ & YM with torsion
2. YM theory & instantons on 8d $Z(G/H)$
   - Quick review of 7d G_2- & 8d Spin(7)-structures
   - Set-up: gauge field ansatz
   - Solutions: old & new
3. Conclusions

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- In the **low-energy limit**, heterotic string theory yields $\mathcal{N} = 1$, $d = 10$ supergravity coupled to super Yang-Mills theory
- In phenomenological applications, one often considers **"string compactifications"**: $\mathcal{M}^{10} = \mathcal{M}^{10-n} \times X_n$
- Of particular interest are solutions that preserve some amount of **supersymmetry**
- Condition of SUSY preservation leads to appearance of higher-dim. **YM-instantons and G-structure manifolds**

Overarching aim(s):

1. **construct new instanton/YM solutions on various G-structure manifolds**
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Definition

A Yang-Mills instanton is a gauge connection$^{*)}$ on Euclidean $\mathcal{M}^4$, whose curvature F is **self-dual**, i.e. $*F = F$.

$^{*)}$connection ${}^A\nabla$ on a principal K-bundle over $\mathcal{M}^4$ (gauge group K)

[Belavin, Polyakov, Schwarz, Tyupkin (1975); Atiyah, Hitchin, Singer (1977); Atiyah, Drinfeld, Hitchin, Manin (1977), . . .]

Properties

- Solutions of YM-eq. $(0 \overset{\text{BI}}{=} DF = D*F \implies D*F = 0)$
- 1st order eq. easier to solve than 2nd order YM-eq.
- 1st ex: **BPST instanton** (1975) for $\mathcal{M} = \mathbb{R}^4$, $K = SU(2)$

Widespread applications in maths & physics

- **classification of 4-manifolds** (e.g. Donaldson invariants)
- learn about **structure of YM-vacuum** (crit. pts. of YM-action; appear in path int. as leading qu. corr.)

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In higher dimensions, the **instanton equation** is generalized to

$$*F = -F \wedge *Q_{\mathcal{M}} \ ,$$

with some globally well-defined 4-form $Q_{\mathcal{M}}$.

[Corrigan, Devchand, Fairlie, Nuyts (1983); Ward (1984), . . .]

Properties

- Need additional structure on $\mathcal{M}$ to have $Q_{\mathcal{M}} \leftrightarrow G$**-structure manifolds** (i.e. struct. grp. $G \subset SO(d)$, e.g. $SU(3)$ in $d = 6$)
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Alternative defs (in many phys. applic.: 3 defs. equivalent)

- $F \cdot \epsilon = 0$ (BPS eq. in string theory)
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Scope of rest of talk

Find explicit solutions of instanton eq. and YM-eq. w/ torsion on **"cylinder"** $Z(G/H) := \mathbb{R} \times G/H$.

- G/H is a 7d compact coset space w/ G_2- **or** $SU(3)$-**structure**
- Cylinder **metric**: $g = \mathrm{d}\tau \otimes \mathrm{d}\tau + \delta_{ab} e^a \otimes e^b$ $(a, b = 1, \ldots, 7)$
- $\{e^\mu\} = \{e^0 = \mathrm{d}\tau, e^a\}$ is a local ONB of $T^*(\mathbb{R} \times G/H)$
- **Why coset spaces?** $\rightarrow$ simple non-triv. examples of G-structure manifolds (eqs. manageable)
- **Why cylinders?** $\rightarrow$ reduce to ODEs (gradient flow eqs.) in τ
- Further **motivation**
  - Soln in gauge sector of **heterotic flux compactifications** (as e.g. in [AH, Lechtenfeld, Musaev (2014)])
  - **Fill a gap** in literature on higher-dim YM instantons [Lechtenfeld, Bauer, Bunk, Geipel, Gemmer, Harland, Ivanova, Lubbe, Nölle, Popov, Rahn, Sperling, Tormählen, AH, ... (2009–...)]

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7d G_2-structures:

- G_2-str. def. by **3-form** P (Hodge dual **4-form** $Q := *_7 P$)
- G_2-structures distinguished/classified by **4 torsion classes**:

$$\mathrm{d}P = \tau_0\, Q + 3\,\tau_1 \wedge P + *_7\tau_3\ , \qquad \mathrm{d}Q = 4\,\tau_1 \wedge Q + \tau_2 \wedge P$$

- Important examples:

Type	TCs	Properties
parallel	$\emptyset$	$\mathrm{d}P = 0$, $\mathrm{d}Q = 0$
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$$\Psi = P \wedge \mathrm{d}\tau - Q$$

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- Back to YM theory on $Z(G/H)$
- **"Natural" G-invariant ansatz** on $Z(G/H)$:

$$A = e^i I_i + e^a X_a(\tau) \qquad \text{(temporal gauge: no } d\tau \text{ term)}$$

[Bauer, Ivanova, Lechtenfeld, Lubbe (2010); ...]

- **Notation**:
  - Lie algebra decomposes: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad (\mathfrak{m} \leftrightarrow G/H)$
  - Lie algebra generators of $\mathfrak{g}$ split: $\{I_A\} = \{I_i\} \cup \{I_a\}$
  - Lie algebra:

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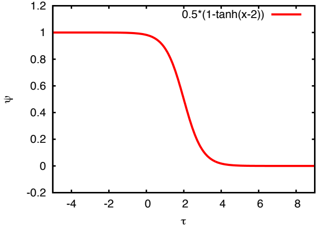

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Other (known) universal YM-solutions:

- Now, consider **YM-eq. w/ torsion** $D * F + F \wedge *H = 0$
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$$\sum_a [X_a, \dot{X}_a] = 0 \qquad \textbf{Gauss-law constraint}$$

$$\ddot{X}_a = \left(\tfrac{1}{2}(f_{acd} - H_{acd})f_{bcd} - f_{aci}f_{bci}\right) X_b - \tfrac{1}{2}(3f_{abc} - H_{abc})[X_b, X_c] - [X_b, [X_b, X_a]] - \tfrac{1}{2} H_{abc} f_{ibc} I_i$$

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Case-by-case analysis:

Consider **multi-field** configurations ...

- ... on cylinders over **three** 7d cosets with nearly parallel G_2-structure
  - Berger space $SO(5)/SO(3)_{\text{max}}$
  - Squashed 7-sphere $Sp(2) \times Sp(1)/Sp(1)^2$
  - (Aloff-Wallach spaces $SU(3)/U(1)_{k,l}$, cf. also [AH, Ivanova, Lechtenfeld, Popov (2011); Geipel (2016)])
- ... and on cylinders over **four** 7d cosets with $SU(3)$-structure ($SU(3) \subset G_2$, special case of G_2-struct.)
  - $(SO(5)/SO(3)_{A+B})$
  - $(N^{pqr} = (SU(3) \times U(1))/(U(1) \times U(1)))$
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Berger space & squashed S^7:

- 1st step to determine multi-field sol:
  solve G-**inv. cond.** $[I_i, X_a] = f_{ia}^{b} X_b$
- **Berger space**: G-inv. cond. $\implies$ $X_a = \phi I_a$
  (back to single field case: **nothing new**)
- **Squashed** S^7: G-inv. cond. $\implies$ 2 real fields ϕ_1, ϕ_2
  - Instanton eq.: $\phi_1 = \pm\phi_2 = \pm\phi$ (again, **back to old case**)
  - YM-eq. $\implies$ **2 branches**:

1. **"instanton branch"** ($\phi_1 = \pm\phi_2 = \pm\phi$)
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2. $\phi_2 = (\kappa + 3)/2$, $\phi_1(\tau) = \pm\sqrt{C_\kappa}\tanh\left[\frac{\sqrt{C_\kappa}}{2}(\tau - \tau_0)\right]$
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Non-trivial multi-field solution I ([AH (2016)]):

- **1st example**: $Z(M^{pqr})$, $M^{pqr} = \frac{SU(3)\times SU(2)\times U(1)}{SU(2)\times U(1)\times U(1)}$
  - $SU(3)$-structure **only** for $p = \pm q$, $r = 0$. W.l.o.g. take M^{110}
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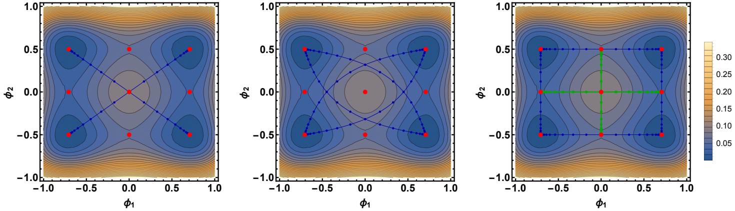

Analytical multi-field solutions (of YM w/ torsion)
Blue: finite-energy (physical) YM-configs. Green: $E \to \infty$.

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3. **Both arise naturally** in S.T. together with G-**structure**
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Open Problems & WIP

- Other cosets, ansätze, corners of param./field space, ...
- Find explicit **S.T. embeddings**. Promising candidate: het SUGRA w/ $\mathbb{R}^{1,1} \times \mathbb{R} \times G/H$ + **domain wall** structure (?) (analog of [AH, Lechtenfeld, Musaev (2014)])

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