Supergravity U-folds and symplectic duality bundles

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Motivation

Supergavity theories are supersymmetric theories of gravity that some of us love, some of us hate, but that sooner or later we all have to use when working in String Theory. There is no doubt that supergravity theories are interesting by themselves as well as because of their relation to String Theory. But... how well do *we* understand supergravity?

- The local structure of the theory has been extensively studied in the literature since the 70's.
- The local structure of all ungauged supergravities is well known and has been classified by dimension and amount supersymmetry preserved.
- The local gaugings of supergravity have also been extensively studied, but there is still a lot of work to be done before arriving to a complete classification which clarifies their possible String/M-theory origin.
- Various formalisms, such as generalized geometry and double field theory, have have been developed to further explore the local structure of supergravity.
- Local supersymmetric (and non-supersymmetric to a lesser extent) solutions of supergravity have been widely studied in the literature.

It looks like there is a well-established program to study the **local!** structure of supergravity and its zoo of solutions.

What about the **GLOBAL** structure of supergravity?

But wait...what does *global structure of supergravity* even mean?

Any proper theory of gravity should be formulated on a (perhaps time-like oriented or oriented) Lorentzian manifold whose physical Lorentzian metric determines the causality of the space-time. Supergravity contains many other fields, aside from the metric, so what about them?

- Obtaining the global structure of supergravity on a differentiable manifold consists of two steps:
  1. Determining its complete matter content in terms of connections and global sections of the appropriate fiber bundles (or more generally, submersions or gerbes) equipped with the appropriate geometric structures.
  2. Determining the global differential operators acting on the corresponding spaces of sections (generically infinite-dimensional Fréchet manifolds) which are required to formulate the equations of motion of the theory.

It turns out that the problem of studying the mathematically rigorous global formulation of supergravity has not been systematically addressed in the literature.

Mathematical motivation

The goal of this project is hence to develop the **mathematical theory of supergravity**. Having a global and complete formulation of supergravity is a necessary step of utmost importance in order to perform a systematic mathematical study of supergravity. For example:

1. Develop spin geometry beyond the standard set-up of spin/spin^c-structures: Lipschitz-structures required.
2. Study supergravity theories as a global system of partial differential equations on Lorentzian manifolds with different causality conditions: generalized supersymmetric flow equations and reduced systems.
3. Study the moduli space of solutions of supergravity theories. Characterize its topological and geometric properties. Global analysis required.
4. Study the automorphism group of supergravity theories.
5. Study the global structure of supergravity solutions: maximal and complete extensions of local Lorentzian solutions.
6. Explore possible applications of supersymmetric PDE's to study the topology and geometry of differentiable manifolds: this generalizes Donaldson's program on the differentiable topology of lower-dimensional manifolds, which is based on instanton equations.

This is clearly mathematically relevant. However, is it also physically relevant? Interestingly enough, understanding the global structure of supergravity clarifies many physical aspects of supergravity and its solutions.

Physical motivation

Understanding the global mathematical structure of supergravity is also physically relevant. In the process of understanding the global mathematical formulation of supergravity we will:

1. find new supergravity theories, namely new non-trivial extensions of local supergravity. Hence, **we are not just rewritting local supergravity using fancy mathematical tools. We are obtaining new physical supersymmetric theories**.
2. be able to give a precise definition of the global U-duality group of a supergravity theory as the automorphism group of the appropriate geometric structures and fiber bundles (for instance, positive polarizations on flat vector bundles).
3. be able to actually *compute* the U-duality group of a supergravity theory, showing that it differs from the naive expectation based on a local analysis.
4. find that generic supergravity solutions are in fact locally geometric U-folds. In addition, we will be able to give a precise definition of locally geometric U-fold in this supergravity context and characterize some of its topological properties.
5. relate the non-triviality of supergravity U-folds to the non-triviality of the fundamental group of the space-time manifold and the scalar manifold.
6. obtain a global and manifiestly U-duality invariant formulation of supergravity.

In order to obtain the global structure of supergravity we need to split the problem in two parts:

1. We first consider the bosonic sector of supergravity, obtaining the global formulation of such sector. This leads to generalized Einstein-Section-Maxwell theories.
2. We then supersymmetrize the previous bosonic theoory, yielding generalized supergravity. For this we need to use the appropriate mathematical framework to work with spinors in supergravity. Such framework is yet not available: it requires a complete understanding of real and complex bundles of Clifford modules over the bundle of Clifford alegebras (and its even part) of the underlying space-time manifold.

Point (2) is very subtle and delicate. Progress has been recently made by C. Lazaroiu and CSS by exploring the theory of real Lipschitz structures. Lipschitz structures were original introduced in the seminal work of T. Friedrich and A. Trautman for bundles of faithful complex Clifford-modules. In this talk we will exclusively consider point (1) in the particular case of four dimensions.

Understanding the global mathematical formulation of the bosonic sector of four-dimensional supergravity boils down to answer the following two fundamental questions:

1. What is a *scalar field* in supergravity?
2. What is a *vector field* and a *field strength* (or rather a bunch of them) in supergravity?

These look like very elementary, even undergraduate-level, questions. However, properly answering them in the context of supergravity requires a remarkably subtle mathematical technology. Standard local four-dimensional supergravity action:

$$\mathcal{L} = \frac{1}{2\kappa}\mathrm{R}(g) + \frac{1}{2}\mathcal{G}_{AB}\partial_\mu\varphi^A\partial^\mu\varphi^B - \gamma_{ij}F^i_{\mu\nu}F^{j\mu\nu} - \theta_{ij}F^i_{\mu\nu}(*_g F^j)^{\mu\nu} + \Phi\,.$$

We have:

- Scalar fields $\varphi^A(x^\mu)$.
- Field strengths $F^i_{\mu\nu}$.
- Scalar-dependent couplings $\mathcal{G}_{AB}(\varphi^A)$, $\gamma_{ij}(\varphi^A)$ and $\theta_{ij}(\varphi^A)$.

All these are local objects! They make sense on a contractible open set of a four-dimensional manifold M.

The answer to the previous two fundamental questions we requires to consider the following three structures:

- A *flat Kaluza Klein space* $\pi\colon (E, h) \to (M, g)$. This is a particular type of surjective pseudo-Riemannian submersion π defined over space-time M, whose total space carries a geodesically complete Lorentzian metric h making the fibers into totally-geodesic Riemannian submanifolds. In particular, π is a fiber bundle endowed with a complete Ehresmann connection whose transport acts through isometries between the fibers.
- A *duality bundle*. This is a flat symplectic vector bundle $\boldsymbol{\Delta} = (\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D})$ defined over the total space E of the previously introduced Kaluza-Klein space.
- A *vertical taming* J on $\boldsymbol{\Delta}$, which defines a complex polarization of the complexified bundle $(\boldsymbol{\mathcal{S}}_{\mathbb{C}}, \boldsymbol{\omega}_{\mathbb{C}})$.

The previous three structures are the basic structures required to formulate bosonic generalized supergravity: if we fix them we completely fix the bosonic sector of the theory.

Vertical tamings

Definition

The *extended horizontal transport* along a path $\gamma \in \mathcal{P}(M)$ is the unbased isomorphism of vector bundles $\mathbf{T}_\gamma : \mathcal{S}_{\gamma(0)} \to \mathcal{S}_{\gamma(1)}$ defined through:

$$\mathbf{T}_\gamma(e) \stackrel{\text{def.}}{=} U_{\gamma_e} : \mathcal{S}_e \to \mathcal{S}_{T_\gamma(e)} \,, \quad \forall\, e \in E_{\gamma(0)} \,,$$

which linearizes the Ehresmann transport $T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ along γ.

Definition

A taming $\mathbf{J}$ of $(\boldsymbol{\mathcal{S}}, \boldsymbol{\omega}, \mathbf{D})$ is called *vertical* if it is $\mathbf{T}$-invariant, which means that it satisfies:

$$\mathbf{T}_\gamma \circ J_{\gamma(0)} = J_{\gamma(1)} \circ \mathbf{T}_\gamma \,, \qquad \forall\, \gamma \in \mathcal{P}(M) \,.$$

It is clear that $\mathbf{J}$ is vertical if and only if it satisfies:

$$\mathbf{D}_X \circ \mathbf{J} = \mathbf{J} \circ \mathbf{D}_X \,, \qquad \forall\, X \in \Gamma(E, H) \,.$$

In this case, $\mathbf{T}_\gamma$ is an isomorphism of tamed flat symplectic vector bundles:

$$\mathbf{T}_\gamma \colon (\mathcal{S}_{\gamma(0)}, D_{\gamma(0)}, \omega_{\gamma(0)}, J_{\gamma(0)}) \xrightarrow{\sim} (\mathcal{S}_{\gamma(1)}, D_{\gamma(1)}, \omega_{\gamma(1)}, J_{\gamma(1)}) \, ,$$

which covers the isometry $T_\gamma : (E_{\gamma(0)}, h_{\gamma(0)}) \to (E_{\gamma(1)}, h_{\gamma(1)})$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{S}_{\gamma(0)}, D_{\gamma(0)}, \omega_{\gamma(0)}, J_{\gamma(0)}) & \xrightarrow{\mathbf{T}_\gamma} & (\mathcal{S}_{\gamma(1)}, D_{\gamma(1)}, \omega_{\gamma(1)}, J_{\gamma(1)}) \\
\Big\downarrow{\scriptstyle \pi_{\gamma(0)}} & & \Big\downarrow{\scriptstyle \pi_{\gamma(1)}} \\
(E_{\gamma(0)}, h_{\gamma(0)}) & \xrightarrow{T_\gamma} & (E_{\gamma(1)}, h_{\gamma(1)})
\end{array}$$

Definition

Let $\pi\colon (E, h) \to (M, g)$ be a Lorentzian submersion. An *electromagnetic bundle* $\mathbf{\Xi}$ is a duality bundle $\mathbf{\Delta}$ over E equipped with a vertical taming $\mathbf{J}$.

Definition

A *scalar-electromagnetic bundle* defined over (M, g) is a triple:

$$\mathcal{D} = (\pi : (E, h) \to (M, g), \bar{\Phi}, \Xi)$$

consisting of a Kaluza-Klein space $\pi : (E, h) \to (M, g)$, a vertical potential $\bar{\Phi}$ and an electromagnetic bundle Ξ defined over the total space E of π. The scalar-electromagnetic bundle $\mathcal{D}$ is called *integrable* if $\pi : (E, h) \to (M, g)$ is an integrable Kaluza-Klein space.

A scalar-electromagnetic bundle is the data we need to specify in order to define the bosonic sector of ungauged supergravity on a four-manifold.

- If we change the underlying scalar-electromagnetic bundle we change the theories that we can define on the space-time manifold M!!

Moduli of duality bundles:

$$\mathcal{M} = \mathrm{Hom}(\pi_1(E), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R})$$

.

Definition

Let $\mathcal{D} = (\pi, \bar{\Phi}, \Xi)$ be a scalar-electromagnetic bundle defined with associated Lorentzian submersion $\pi\colon (E, h) \to (M, g)$ and let $s \in \Gamma(\pi)$ be a section of π. An *electromagnetic field strength* is a two-form $\boldsymbol{\mathcal{V}} \in \Omega^2(M, \boldsymbol{\mathcal{S}}^s)$ having the following properties:

1. $\boldsymbol{\mathcal{V}}$ is positively-polarized with respect to $\mathbf{J}^s$, i.e. the following relation is satisfied:

$$*_g \boldsymbol{\mathcal{V}} = -\mathbf{J}^s \boldsymbol{\mathcal{V}}\,.$$

2. $\boldsymbol{\mathcal{V}}$ satisfies the *electromagnetic equation* with respect to s:

$$\mathrm{d}_{\mathbf{D}^s} \boldsymbol{\mathcal{V}} = 0 \quad .$$

Once a choice of scalar-electromagnetic bundle has been made, the matter content of a generalized Einstein-Section-Maxwell theory is given by:

- A Lorentzian metric g on M.
- A section $s \in \Gamma(\pi)$ of the flat Kaluza-Klein space $\pi\colon (E, h) \to (M, g)$.
- A positively-polarized electromagnetic field strenght $\boldsymbol{\mathcal{V}} \in \Omega^2(M, \boldsymbol{\mathcal{S}}^s)$.

This defines the configuration space of the theory: crucial for understanding moduli spaces!

Definition

The *twisted exterior pairing* $(\ ,\ ) := (\ ,\ )_{g,\boldsymbol{Q}^s}$ is the unique pseudo-Euclidean scalar product on the twisted exterior bundle $\wedge_M(\boldsymbol{\mathcal{S}}^s)$ which satisfies:

$$(\rho_1 \otimes \xi_1, \rho_2 \otimes \xi_2)_{g,\boldsymbol{Q}^s} = (\rho_1, \rho_2)_g \boldsymbol{Q}^s(\xi_1, \xi_2)\,,$$

for any $\rho_1, \rho_2 \in \Omega(M)$ and any $\xi_1, \xi_2 \in \Omega^0(M, \boldsymbol{\mathcal{S}}^s)$. Here $\boldsymbol{Q}(\xi_1, \xi_2) = \boldsymbol{\omega}(\mathbf{J}\xi_1, \xi_2)$ and the superscript denotes pull-back by s.

For any vector bundle W, we trivially extend the twisted exterior pairing to a W-valued pairing, which for simplicity we denote by the same symbol, between the bundles $W \otimes (\wedge_M(\mathcal{S}^\varphi))$ and $\wedge_M(\mathcal{S}^\varphi)$. Thus:

$$(e \otimes \eta_1, \eta_2)_{g,\boldsymbol{Q}^s} \stackrel{\text{def.}}{=} e \otimes (\eta_1, \eta_2)_{g,\boldsymbol{Q}^s}\,, \quad \forall\, e \in \Omega^0(M, W)\,, \quad \forall\, \eta_1, \eta_2 \in \wedge_M(\boldsymbol{\mathcal{S}}^s)\,.$$

The *inner g-contraction of two-tensors* is the bundle morphism $\oslash_g : (\otimes^2 T^* M)^{\otimes 2} \to \otimes^2 T^* M$ uniquely determined by the condition:

$$(\alpha_1 \otimes \alpha_2) \oslash_g (\alpha_3 \otimes \alpha_4) = (\alpha_2, \alpha_3)_g \alpha_1 \otimes \alpha_4 \,, \quad \forall\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Omega^1(M) \,.$$

We define the *inner g-contraction of two-forms* to be the restriction of $\oslash_g$ to $\wedge^2 T^* M \otimes \wedge^2 T^* M \stackrel{\oslash_g}{\to} \otimes^2 T^* M$.

Definition

We define the *twisted inner contraction* of $\boldsymbol{\mathcal{S}}^s$-valued two-forms to be the unique morphism of vector bundles $\oslash\colon\ \wedge^2_M (\boldsymbol{\mathcal{S}}^s) \times_M \wedge^2_M(\boldsymbol{\mathcal{S}}^s) \to \otimes^2(T^* M)$ which satisfies:

$$(\rho_1 \otimes \xi_1) \oslash (\rho_2 \otimes \xi_2) = \boldsymbol{Q}^s(\xi_1, \xi_2)\rho_1 \oslash_g \rho_2 \,,$$

for all $\rho_1, \rho_2 \in \Omega^2(M)$ and all $\xi_2, \xi_2 \in \Omega^0(M, \boldsymbol{\mathcal{S}}^s)$.

- The twisted inner contraction is necessary in order to globally write the equations of motion of supergravity!

Let $\mathcal{D} = (\pi, \bar{\Phi}, \Xi)$ be a scalar-electromagnetic bundle with associated electromagnetic bundle $\Xi = (\mathcal{S}, \omega, \mathbf{D}, \mathbf{J})$ and Lorentzian submersion $\pi\colon (E, h) \to (M, g)$. Let:

$$\mathbf{D}^{\mathrm{ad}}\colon \Omega^0(E, End(\mathcal{S})) \to \Omega^1(E, End(\mathcal{S}))\,,$$

be the connection induced by $\mathbf{D}$ on the endomorphism bundle $End(\mathcal{S})$ of $\mathcal{S}$.

Definition

The *fundamental bundle form* Θ associated to $\mathcal{D}$ is the $End(\mathcal{S})$-valued one-form defined on E as follows:

$$\Theta \stackrel{\mathrm{def.}}{=} \mathbf{D}^{\mathrm{ad}}\mathbf{J} \in \Omega^0(E, V^* \otimes End(\mathcal{S}))\,.$$

Definition

The *fundamental bundle field* Ψ associated to $\mathcal{D}$ is the $End(\mathcal{S})$-valued vector field defined on E as follows:

$$\Psi \stackrel{\mathrm{def.}}{=} \left(\sharp_h \otimes \mathrm{Id}_{End(\mathcal{S})}\right) \circ D^{\mathrm{ad}}\mathbf{J} \in \Omega^0(E, V \otimes End(\mathcal{S}))\,.$$

Definition

The *vertical Lagrange density* of π is the functional $e^v_\Phi : \Gamma(\pi) \to \mathcal{C}^\infty(M, \mathbb{R})$ defined, for every $s \in \Gamma(\pi)$, as follows:

$$S_{\rm sc}[g,h,s] = -\int_U \nu_M(g) e^v_\Phi(g,h,s)\,, \qquad e^v_\Phi(g,h,s) \stackrel{\rm def.}{=} \frac{1}{2}{\rm Tr}_g s^*(h_V) + \bar{\Phi}^s \quad .$$

Here $\bar{\Phi}^s = \bar{\Phi} \circ s \in \mathcal{C}^\infty(M, \mathbb{R})$.

Let $s \in \Gamma(\pi)$ be a section. The differential ${\rm d}s \colon TM \to TE$ of s is an unbased morphism of vector bundles, which is equivalent to a section ${\rm d}s \in \Omega^1(M, TE^s)$ which for simplicity we denote by the same symbol. We define $d^v s \stackrel{\rm def.}{=} P_V \circ {\rm d}s \in \Omega^1(M, V^s)$. The Levi-Civita connection on (M, g) together with the s-pull-back of the connection ∇^v on V induce a connection on $T^*M \otimes V^s$, which we denote again by ∇^v.

Definition

The *vertical tension field* of $s \in \Gamma(\pi)$ is defined through:

$$\tau^v(g,h,s) \stackrel{\rm def.}{=} {\rm Tr}_g \nabla^v {\rm d}^v s \in \Gamma(M, V^s) \quad .$$

Definition

Let $\mathcal{D} = (\pi, \bar{\Phi}, \Xi)$ be a scalar-electromagnetic bundle with associated Lorentzian submersion $\pi\colon (E, h) \to (M, g)$. A GESM-theory associated to $\mathcal{D}$ is defined by the following set of partial differential equations on (M, g):

- The Einstein equations:

$$\mathcal{E}_E(g, s, \mathcal{V}) \stackrel{\text{def.}}{=} \mathrm{G}(g) - \kappa\, \mathrm{T}(g, s, \mathcal{V}) = 0\,, \tag{1}$$

  where $\mathrm{T}(g, s, \mathcal{V}) \in \Omega^0(S^2 T^* M)$ is the energy-momentum tensor of the theory, which is given by:

$$\mathrm{T}(g, s, \mathcal{V}) = g\, e_0^v(g, , h, s) - h^s + \frac{g}{2}\bar{\Phi}^{\varphi} + 2\, \mathcal{V} \oslash \mathcal{V}\,.$$

- The scalar equations:

$$\mathcal{E}_S(g, s, \mathcal{V}) \stackrel{\text{def.}}{=} \tau^v(g, h, s) + (\mathrm{grad}_h \bar{\Phi})^s - \frac{1}{2}(*\mathcal{V}, \Psi^s \mathcal{V}) = 0\,. \tag{2}$$

- The electromagnetic (or Maxwell) equations:

$$\mathcal{E}_K(g, s, \mathcal{V}) \stackrel{\text{def.}}{=} \mathrm{d}_{\mathbf{D}^s} \mathcal{V} = 0\,. \tag{3}$$

with unknowns given by triples $(g, s, \mathcal{V}) \in \mathrm{Conf}_{\mathcal{D}}(M)$.

Let $(\pi : (E, h) \to (M, g), \bar{\Phi})$ be an *integrable* bundle of scalar data of type $(\mathcal{M}, \mathcal{G}, \Phi)$. We consider a special trivializing atlas of π defined by the geodesically convex open sets $(U_\alpha)_{\alpha \in I}$ (which cover M). Since π is integrable, the appropriate trivializing maps give isometries $q_\alpha : (E_\alpha, h_\alpha) \xrightarrow{\sim} (U_\alpha \times \mathcal{M}, g_\alpha \times \mathcal{G})$. For any pair of indices $\alpha, \beta \in I$ such that $U_{\alpha\beta} \stackrel{\text{def.}}{=} U_\alpha \cap U_\beta$ is non-empty, the composition $q_{\alpha\beta} \stackrel{\text{def.}}{=} q_\beta \circ q_\alpha^{-1} : U_{\alpha\beta} \times \mathcal{M} \to U_{\alpha\beta} \times \mathcal{M}$ has the form $q_{\alpha\beta}(m, p) = (m, \mathbf{g}_{\alpha\beta}(p))$, where:

$$\mathbf{g}_{\alpha\beta} \in \mathrm{Iso}(\mathcal{M}, \mathcal{G}, \Phi)\,.$$

Setting $\mathbf{g}_{\alpha\beta} = \mathrm{id}_{\mathcal{M}}$ for $U_{\alpha\beta} = \emptyset$, the collection $(\mathbf{g}_{\alpha\beta})_{\alpha,\beta\in I}$ satisfies the cocycle condition:

$$\mathbf{g}_{\beta\delta}\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\delta}\,, \quad \forall\, \alpha, \beta, \delta \in I\,. \tag{4}$$

For any section $s \in \Gamma(\pi)$, the restriction $s_\alpha \stackrel{\text{def.}}{=} s|_{U_\alpha}$ corresponds through q_α to the graph $\mathrm{graph}(\varphi_\alpha) \in \Gamma(\pi_\alpha^0)$ of a uniquely-defined smooth map $\varphi_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$:

$$s_\alpha = q_\alpha^{-1} \circ \mathrm{graph}(\varphi_\alpha) \;\; \text{i.e.} \;\; q_\alpha(s_\alpha(m)) = (m, \varphi_\alpha(m))\,, \;\; \forall m \in U_\alpha\,. \tag{5}$$

Composing the first relation from the left with p_0^α gives:

$$\varphi^\alpha = \hat{q}_\alpha \circ s_\alpha \quad .$$

Which in turn implies:

$$\varphi^\beta(m) = \mathbf{g}_{\alpha\beta}\varphi^\alpha(m) \ \ \forall m \in U_{\alpha\beta} \quad , \tag{6}$$

where juxtaposition in the right hand side denotes the tautological action of the group $\mathrm{Isom}(\mathcal{M}, \mathcal{G}, \Phi)$ on $\mathcal{M}$. Conversely, any family of smooth maps $\{\varphi^\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})\}_{\alpha\in I}$ satisfying (6) defines a smooth section $s \in \Gamma(\pi)$ whose restrictions to U_α are given by (5). The equation of motion for s is equivalent with the condition that each φ^α satisfies the equation of motion of the ordinary sigma model defined by the scalar data $(\mathcal{M}, \mathcal{G}, \Phi)$ on the space-time (U_α, g_α):

$$\tau^v(h, s) = -(\mathrm{grad}\bar{\Phi})^s \Leftrightarrow \tau(g_\alpha, \varphi^\alpha) = -(\mathrm{grad}\Phi)^{\varphi^\alpha} \ \ \forall \alpha \in I \quad . \tag{7}$$

Thus global solutions s of the equations of motion are *glued* from local solutions $\varphi^\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$ of the equations of motion of the ordinary sigma model using the $\mathrm{Iso}(\mathcal{M}, \mathcal{G}, \Phi)$-valued constant transition functions $\mathbf{g}_{\alpha\beta}$ which satisfy the cocycle condition (4).

Let $\boldsymbol{\mathcal{D}} = (\pi, \bar{\Phi}, \Xi)$ be an *integrable* scalar-electromagnetic bundle and let $(U_\alpha, \mathbf{q}_\alpha)_{\alpha\in I}$ be a special trivializing atlas. Let $s \in \Gamma(\pi)$ and and $\mathcal{V} \in \Omega^2(M, \boldsymbol{\mathcal{S}}^s)$. Let $s_\alpha \stackrel{\text{def.}}{=} s|_{U_\alpha}$ and $\mathcal{V}_\alpha \stackrel{\text{def.}}{=} \mathcal{V}|_{U_\alpha} \in \Omega^2(U_\alpha, \boldsymbol{\mathcal{S}}^s)$. The diffeomorphisms $q_\alpha : E_\alpha \to E^0_\alpha = U_\alpha \times \mathcal{M}$ and their linearizations $\mathbf{q}_\alpha : \boldsymbol{\mathcal{S}}_\alpha \to \boldsymbol{\mathcal{S}}^0_\alpha \stackrel{\text{def.}}{=} \mathcal{S}^{p^0_\alpha}$ identify s_α with maps $\varphi_\alpha \stackrel{\text{def.}}{=} p^0_\alpha \circ q_\alpha(s_\alpha) \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$. The isomorphism of electromagnetic structures $\mathbf{q}_\alpha : (\boldsymbol{\mathcal{S}}_\alpha, \boldsymbol{\omega}_\alpha, \mathbf{D}_\alpha, \mathbf{J}_\alpha) \stackrel{\sim}{\to} (\boldsymbol{\mathcal{S}}^0_\alpha, \boldsymbol{\omega}^0_\alpha, \mathbf{D}^0_\alpha, \mathbf{J}^0_\alpha)$ pulls back to an isomorphism:

$$\mathbf{q}^s_\alpha : (\boldsymbol{\mathcal{S}}^s_\alpha, \boldsymbol{\omega}^s_\alpha, \mathbf{D}^s_\alpha, \mathbf{J}^s_\alpha) \stackrel{\sim}{\to} (\mathcal{S}^{\varphi^\alpha}, \omega^{\varphi^\alpha}, D^{\varphi^\alpha}, J^{\varphi^\alpha}) \,,$$

This isomorphism identifies $\boldsymbol{\mathcal{V}}_\alpha$ with a $\mathcal{S}^{\varphi^\alpha}$-valued two-form defined on U_α through:

$$\mathcal{V}^\alpha \stackrel{\text{def.}}{=} \mathbf{q}^s_\alpha \circ \boldsymbol{\mathcal{V}}_\alpha \in \Omega^2(U_\alpha, \mathcal{S}^{\varphi^\alpha}) \,, \tag{8}$$

and we have:

$$\boldsymbol{\mathcal{V}}_\alpha \in \Omega^{2+,s}_{g_\alpha, \boldsymbol{\mathcal{S}}^s_\alpha, \mathbf{J}^s_\alpha}(U_\alpha) \quad \text{iff} \quad \mathcal{V}^\alpha \in \Omega^{2+,\varphi^\alpha}_{g_\alpha, \mathcal{S}^{\varphi^\alpha}, J^{\varphi^\alpha}}(U_\alpha)$$
$$\mathrm{d}_{\boldsymbol{\mathcal{D}}^s} \boldsymbol{\mathcal{V}}_\alpha = 0 \quad \text{iff} \quad \mathrm{d}_{D^{\varphi^\alpha}} \mathcal{V}^\alpha = 0 \,.$$

$$(s, \boldsymbol{\mathcal{V}}) \in \mathrm{Sol}^g_{\boldsymbol{\mathcal{D}}}(M) \;\; \text{iff} \;\; (\varphi^\alpha, \mathcal{V}^\alpha) \in \mathrm{Sol}^{g_\alpha}_{\mathcal{D}}(U_\alpha) \;\; \forall \alpha \in I \quad .$$

Relations (8) imply the gluing conditions:

$$\mathcal{V}^{\beta}|_{U_{\alpha\beta}} = \mathbf{f}^{s}_{\alpha\beta}\mathcal{V}^{\alpha}|_{U_{\alpha\beta}} \quad , \tag{9}$$

which accompany the gluing conditions (6) for φ^{α}.
Conversely, any pair of families $(\varphi_{\alpha})_{\alpha\in I}$ and $(\mathcal{V}^{\alpha})_{\alpha\in I}$ of solutions of the equations of motion of the ordinary scalar sigma model with Abelian gauge fields of type $\mathcal{D} = (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, \omega, D, J)$ defined on the open sets $U_{\alpha} \subset M$ of a special open cover for (M, g) which satisfy conditions (6) and (9) corresponds to a global solution $(s, \mathcal{V})$ of the equations of motion of the section sigma model coupled to Abelian gauge fields defined by the scalar-electromagnetic bundle $\boldsymbol{\mathcal{D}}$. Hence such global solutions are obtained by gluing local solutions of the ordinary sigma model coupled to Abelian gauge fields using scalar-electromagnetic symmetries of the latter.

Definition

Let $\mathcal{D}$ be a scalar-electromagnetic structure. A classical locally-geometric U-fold of type $\mathcal{D}$ is a global solution $(g, s, \boldsymbol{\mathcal{V}}) \in \mathrm{Sol}_{\boldsymbol{\mathcal{D}}}(M)$ of the equations of motion a GESM theory associated to an scalar-electromagnetic bundle $\boldsymbol{\mathcal{D}}$ of type $\mathcal{D}$ with integrable associated Lorentzian submersion.

Let $\mathcal{D} \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, D, \omega, J)$.

Definition

The *scalar-electromagnetic symmetry group* of $\mathcal{D}$ is the subgroup $\mathrm{Aut}(\mathcal{D})$ of $\mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D, J, \omega)$ defined through:

$$\mathrm{Aut}(\mathcal{D}) \stackrel{\text{def.}}{=} \{f \in \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D, J, \omega) \mid f \in \mathrm{Aut}(\mathcal{M}, \mathcal{G}, \Phi)\}\,.$$

An element of this group is called a *scalar-electromagnetic symmetry*.

We have:

$$\mathrm{Aut}(\mathcal{D}) \stackrel{\text{def.}}{=} \{f \in \mathrm{Aut}(\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, D, \omega) \mid \mathbf{Ad}(f)(J) = J\}\,.$$

Theorem

For all $f \in \mathrm{Aut}(\mathcal{D})$, we have:

$$f \diamond \mathrm{Sol}^{g}_{\mathcal{D}}(M) = \mathrm{Sol}^{g}_{\mathcal{D}}(M) \quad .$$

Thus $\mathrm{Aut}(\mathcal{D})$ consists of symmetries of the equations of motion of a GESM theory.

It is natural then to define the U-duality group of a GESM theory as its scalar-electromagnetic symmetry group!

Theorem

Let $\Sigma \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi)$, $\Delta \stackrel{\text{def.}}{=} (\mathcal{S}, D, \omega)$, $\Xi \stackrel{\text{def.}}{=} (\mathcal{S}, D, J, \omega)$ *and* $\mathcal{D}_0 \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, D, \omega)$. *We have short exact sequences:*

$$1 \to \mathrm{Aut}(\Delta) \hookrightarrow \mathrm{Aut}(\mathcal{D}_0) \longrightarrow \mathrm{Aut}^{\Delta}(\Sigma) \to 1$$
$$1 \to \mathrm{Aut}(\Xi) \hookrightarrow \mathrm{Aut}(\mathcal{D}) \longrightarrow \mathrm{Aut}^{\Xi}(\Sigma) \to 1 \quad ,$$

where $\mathrm{Aut}(\Delta)$ *and* $\mathrm{Aut}(\Xi)$ *are the groups of based symmetries of* Δ *and* Ξ. *The groups appearing in the right hand side consist of those automorphisms of the scalar structure* Σ *which respectively admit lifts to scalar-electromagnetic dualities of* $\mathcal{D}_0 = (\Sigma, \Delta)$ *and scalar-electromagnetic symmetries of* $\mathcal{D} = (\Sigma, \Xi)$. *Fixing a point* $p \in \mathcal{M}$, *we can identify* $\mathrm{Aut}(\Delta)$ *with the commutant of* Hol_D^p *inside the group* $\mathrm{Aut}(\mathcal{S}_p, \omega_p) \simeq \mathrm{Sp}(2n, \mathbb{R})$. *In particular, the exact sequences above show that* $\mathrm{Aut}(\mathcal{D}_0)$ *and* $\mathrm{Aut}(\mathcal{D})$ *are Lie groups.*

Further directions

1. Characterize the U-duality group of a general GESM theory and compute it in explicit important cases: spectral sequence expected.
2. Implement Dirac quantization on a general GESM theory.
3. Supersymmetrize GESM theories $\Rightarrow$ Generalized supergravities.
4. Obtain explicit solutions of a non-trivial GESM theory: these would be locally geometric U-folds in four dimensions.
5. Study the higher-dimensional origin of GESM theories by using the appropriate notion of Scherk-Schwarz reduction.
6. Give a global and mathematically rigorous formulation of the gauging of a GESM theory and implement it.
7. Characterize all the supersymmetric solutions of generalized supergravity.
8. Systematic study of GESM and generalized supergravity on globally hyperbolic Lorentzian manifolds. Mathematical study of the type of partial differential equations and flow equations obtained in this way.
9. Study of the moduli space of supersymmetric solutions of generalized supergravity. Possible applications to the study of the geometry and topology of the underlying differentiable manifold.

Thanks!