Superintegrability of Calogero model with oscillator and Coulomb potentials and their generalizations to (pseudo)spheres: Observation

Armen Nersessian
Yerevan State University

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Whole non-triviality of Calogero model (and of other conformal mechanics) is encoded in "spherical part".

With the aid of that "spherical part" I will:

- Construct the analog of Calogero-oscillator and Calogero-Coulomb systems on spheres and hyperboloids,
- Demonstrate their classical superintegrability.
- As a bi-product, I’ll show that the so-called Tremblay-Turbiner-Winternits and Post-Winternitz models (Calogero-oscillator and Calogero-Coulomb systems related with positive roots of Coxeter system $I_2(k)$), are connected to each other by Bohlin transformation.
Content

- Spherical part of conformal mechanics
- Spherical part of Calogero model
- Spherical Calogero in terms of action variables
- Superintegrable Calogero-oscillator and Calogero-Coulomb systems
- Tremblay-Turbiner-Winternitz and Post-Winternitz systems and their relation
Conformal mechanics

\[ \{ H, D \} = 2H, \quad \{ K, D \} = -2K, \quad \{ H, K \} = D \]

Most known example

\[ \omega = dp \wedge dr, \quad \mathcal{H} = \frac{p^2}{2} + V(r), \quad \text{where} \quad r \cdot \nabla V(r) = -2V(r). \]

\[ D = p \cdot r, \quad K = \frac{r^2}{2} \]

The phase space of conformal mechanics is noncompact one: we can't formulate the system in action-angle variables.
Canonical basis and its "spherical part"

\[ SO(1,2) \text{ Casimir : } \mathcal{I} = 2KH - \frac{1}{2}D^2 \]

Introduce radial coordinate and momentum

\[ \mathcal{D} \equiv \rho_r r, \quad \mathcal{K} \equiv \frac{r^2}{2} : \{ P_r, \mathcal{I} \} = \{ r, \mathcal{I} \} \]

\[ \mathcal{H} = \frac{\rho_r^2}{2} + \frac{\mathcal{I}}{r^2} : \{ H, \mathcal{I} \} = 0 \]

One can choose the appropriate "spherical coordinates",

\[ u^A = (\rho_\alpha, \phi^\alpha) : \quad \{ \rho_r, u^A \} = \{ r, u^A \} = 0, \quad \{ \rho_\beta, \phi^\alpha \} = \delta^\alpha_\beta, \]

separating the "radial" and spherical parts of the system.

Thus, we can extract the compact ("spherical") part of conformal mechanics:

\[ (\mathcal{I}(u), \, d\rho_\alpha \wedge d\phi^\alpha) \]
Importance of spherical part

- Splitting of noncompact motion from compact one
- Straightforward application of the textbook scattering theory
- $\mathcal{N} = 4$ superconformal extensions
- (Super)Integrable spherical parts allows to construct new (super)integrable (deformations of known) systems
Superconformal extensions

Let the spherical spherical part of conformal mechanics admits $\mathcal{N} = 4$ supersymmetric extension $\{\Theta^{a\alpha}, \Theta^{b\beta}\} = 2\iota\varepsilon^{ab}\varepsilon^{\alpha\beta}\mathcal{I}_{SUSY}$. Then we can immediately construct $\mathcal{N} = 4$ D(1.2; $\alpha$) superconformal extension of the whole system (T.Hakobyan, O.Lechtenfeld, S.Krivonos, A.N.)

\[
S^{a\alpha} = r\eta^{a\alpha}, \quad Q^{a\alpha} = p_r\eta^{a\alpha} + \frac{\Theta^{a\alpha}}{r} + \ldots,
\]

\[
\mathcal{H}_{SUSY} = \frac{p_r^2}{2} + \frac{\mathcal{I}_{SUSY}}{r^2} - \frac{\iota\Theta^{a\alpha}\eta_{a\alpha}}{r^2} + \ldots
\]
Integrable deformations of $N$-dimensional Models

$\mathbb{IR}^N : \mathcal{H} = \frac{p_r^2}{2} + \frac{\mathcal{I}_{N-1}(I_i)}{r^2} + V(r), \quad V(r) = \begin{cases} \frac{\omega^2 r^2}{2} \\ -\gamma/r \end{cases}$

$S^N : \mathcal{H} = \frac{p_{\chi}^2}{2r_0^2} + \frac{\mathcal{I}_{N-1}}{r_0^2 \sin^2 \chi} + V(\tan \chi), \quad V(r) = \begin{cases} \frac{r_0^2 \omega^2 \tan^2 \chi}{2} \\ -\left(\frac{\gamma \cot \chi}{r_0}\right) \end{cases}$

$H^N : \mathcal{H} = \frac{p_{\chi}^2}{2r_0^2} + \frac{\mathcal{I}_{N-1}}{r_0^2 \sinh^2 \chi} + V(\tanh \chi), \quad V(r) = \begin{cases} \frac{r_0^2 \omega^2 \tanh^2 \chi}{2} \\ -\left(\frac{\gamma \coth \chi}{r_0}\right) \end{cases}$
Quantum and classical spectra

Quantum

\[ E(n_r, l) \rightarrow E(n_r, \mathcal{E}_{n_1, \ldots, n_{N-1}}), \]

where \( \mathcal{E}_{n_1, \ldots, n_{N-1}} \) is spectrum of spherical part.

"Classical"

\[ \mathcal{H}(l_r, l_1 + l_2 + \ldots + l_{N-1}) \rightarrow \mathcal{H}(l_r, \mathcal{I}(l_1, \ldots, l_{N-1})), \]

where \( \mathcal{I}(l_1, \ldots, l_{N-1}) \) is spherical part written in action variables.
How to find hidden symmetries?

Let we have Hamiltonian system formulated in action angle variables, with the Hamiltonian

$$H = H(k_1 l_1 + k_2 l_2 + \ldots + k_K l_K, l_{K+1}, \ldots, l_N)$$

where $k_i$ are integers.

Then the system possesses hidden symmetries given by the functions

$$l_{ij} = \cos(k_j \Phi_i - k_i \Phi_j), \quad i, j = 1, \ldots, K.$$  

Maximally superintegrable systems

$$H = H(k_1 l_1 + k_2 l_2 + \ldots + k_N l_N).$$
"Classical spectra" of deformed oscillator and Coulomb systems

Oscillator

\[ H_{osc} = \begin{cases} 
\omega (2l_r + \sqrt{2I}) & \text{for } \mathbb{IR}^2 \\
\frac{1}{2} (2I\chi + \sqrt{2I} + \omega)^2 - \frac{\omega^2}{2} & \text{for } \mathbb{S}^2 \\
-\frac{1}{2} (2I\chi + \sqrt{2I} - \omega)^2 + \frac{\omega^2}{2} & \text{for } \mathbb{H}^2 
\end{cases} \]

Coulomb

\[ H_C = \begin{cases} 
-\gamma^2 / 2(l_r + \sqrt{2I})^2 & \text{for } \mathbb{IR}^2 \\
-\gamma^2 / 2(l\chi + \sqrt{2I})^2 + (l\chi + \sqrt{2I})^2 / 2 & \text{for } \mathbb{S}^2 \\
-\gamma^2 / 2 \left( l\chi + \sqrt{2I} \right)^2 - (l\chi + \sqrt{2I})^2 / 2 & \text{for } \mathbb{H}^2 
\end{cases} \]
Superintegrable deformations of oscillator and Coulomb systems requires:

\[ I_{Sph} = \frac{1}{2} \left( \sum_{i} k_i l_i \right)^2, \quad k_i \in \mathcal{N} \]

Which systems on spheres have such Hamiltonian?
Simplest example:

\[ I_{MP} = \frac{L^2}{2} + \sum_{i=1}^{N} \frac{g_i^2}{x_i^2}, \quad \sum_{i=1}^{N} x_i^2 = 1. \tag{0.1} \]

It defines nontrivial part of spherical mechanics of a particle in \((2n + 1)\)-dimensional extreme Myers-perry black hole

"Classical Spectrum":

\[ I_{MP} = \frac{1}{2} \left( \sum_{i=1}^{N-1} l_i + \sum_{i=1}^{N} g_i \right)^2 \]
Other example:

Let us define the segment on \((N - 1)\)-dimensional sphere restricting the range of the angles by:

\[ \varphi_i \in [0, 2\pi/k_i), \quad k_i \in \mathbb{Z}. \]

We can immediately find, that its classical spectrum is of required form. However, the configuration space of resulting system is \(N\)-dimensional cone!
The study of spherical part of CM at classical level has been done in our papers with T.Hakobyan, S.Krivonos, O.Lechtenfeld, A.Saghatelian, V.Yeghikyan.

- The spherical part of $N$-particle Calogero model defines multi-center Higgs oscillator on $S^{N-1}$. For example, for $A_3$-CM force centers are located at the vertexes of *cuboctahedron*.

- Spherical parts of rational CM's are maximally superintegrable systems. They allows immediately find hidden symmetries of rational CM

Quantum properties of the spherical part of rational Calogero models were considered in the recent paper of M.Feigin, O.Lechtenfeld, A.Polychronakos.
”Spherical part of conventional CM: from Quantum to Classical”

Quantum

\[ \mathcal{E}_{sph\text{Cal}} = \frac{1}{2} q (q + g + \hbar(N - 2)), \]

where

\[ q \equiv \frac{N(N - 1)}{2} g + \hbar k_1 + 3 \hbar k_3 + \ldots + N \hbar k_N, \]

with all \( k_i \) take nonnegative integer values.

Classical limit

\[ I_{sph\text{Cal}} = \frac{1}{2} \left( \frac{N(N - 1) + 1}{2} g + l_1 + 3 l_3 + \ldots + N l_N + N - 2 \right)^2 \]
Calogero-Coulomb and Calogero oscillator models on $\mathbb{R}^n$

$$\mathcal{H}_{CC} = \frac{p^2}{2} + \sum_i \frac{g^2}{(x_i - x_j)^2} + \frac{\omega^2 x^2}{2}$$

Well-known system.

$$\mathcal{H}_{CC} = \frac{p^2}{2} + \sum_i \frac{g^2}{(x_i - x_j)^2} - \frac{\gamma}{|x|}$$


MAXIMAL SUPERINTEGRABILITY !!!
Generalization to spheres and hyperboloids

\[ V^{(p)s}_{\text{Coulomb}} = \sum_{\alpha \in \Delta_+} \frac{g_{\alpha}^2(\alpha \cdot \alpha) r_0^2}{2(\alpha \cdot \mathbf{x})^2} - \frac{\gamma x_0}{r_0 |\mathbf{x}|}, \quad x_0^2 \pm \mathbf{x}^2 = r_0^2, \]

\[ V^{(p)s}_{\text{osc}} = \sum_{\alpha \in \Delta_+} \frac{g_{\alpha}^2(\alpha \cdot \alpha) r_0^2}{2(\alpha \cdot \mathbf{x})^2} + \frac{\omega^2 r_0^2}{2} \frac{x^2}{x_0^2}, \quad x_0^2 \pm \mathbf{x}^2 = r_0^2. \]

The upper sign corresponds to the sphere, and lower sign – to the hyperboloid.

MAXIMAL SUPERINTEGRABILITY !!!
CO(scillator) - CC(oulomb) correspondence. 2-particle case

d = 2, 3, 5 Coulomb system can be with d = 2, 4, 8-oscillator via Hopf maps. Similar relation holds for respective Calogero-Coulomb and Calogero-Oscillator systems.

\[ H_{TTW} = \pi \bar{\pi} + \frac{f((z/\bar{z})^k)}{z\bar{z}} + \omega^2 z\bar{z}, \]

where \( z = x^1 + ix^2 = re^{\varphi}, f(\varphi) = k^2 \alpha^2 \sin^{-2} k\varphi + k^2 \beta^2 \cos^{-2} k\varphi. \)

Performing canonical (Levi-Civita) transformation \( w = z^2, \quad p = \frac{\pi}{2z}, \) we get

\[ H_{TTW} = 4|w|p\bar{p} + \frac{f(\sqrt{|w|/|\bar{w}|})}{|w|} + \omega^2 |w|. \]
Fixing the level surface $\mathcal{H}_{TTW} = E_{TTW}$, we get

$$\mathcal{H}_{PW} = \mathcal{E}_{PW}, \quad \mathcal{H}_{PW} \equiv p\bar{p} + \frac{f(\sqrt{w/\bar{w}})}{4w\bar{w}} - \frac{\gamma}{|w|}:$$

where

$$\gamma = \frac{E_{TTW}}{4}, \quad \mathcal{E}_{PW} = -\frac{\omega^2}{4}.$$

Generalization of this transformation to the sphere and pseudosphere is straightforward (A.N., G.Pogosyan, 2000). For the higher dimensions the relation between CO- and CC-systems is less trivial.
Thank you for your attention!